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Quadratic functors and metastable homotopy

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For the fiftieth birthday of Steve Halperin

Abstract

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Quadratic functors lead to the fundamental notion of a quadratic $\mathcal{R}$-module M where $\mathcal{R}$ is a ringoid or a ring. We introduce the quadratic tensor product $A \otimes_{\mathcal{R}} M$ and the corresponding abelian group $\mathrm{Hom}_{\mathcal{R}}(A, M)$ consisting of quadratic forms. Then we describe new quadratic derived functors of $\otimes$ and Hom together with applications for homotopy groups of Moore spaces and (co)homology groups of Eilenberg–Mac Lane spaces.

Introduction

In this paper we develop the quadratic homological algebra which is needed for the metastable range of homotopy theory. On the one hand we study quadratic functors and their derived functors (Sections 1–7 and Appendices A and B); on the other hand we describe applications in homotopy theory (Sections 8–11).

Let $\mathcal{Add}(\mathcal{R})$ be the additive completion of a ringoid $\mathcal{R}$ and let $\mathcal{Ab}$ be the category of abelian groups. We classify quadratic functors by "quadratic $\mathcal{R}$-modules"; see Definition 3.1.

Theorem 3.6. *There is a 1–1 correspondence which carries a quadratic functor $F: \mathcal{Add}(\mathcal{R}) \to \mathcal{Ab}$ to a quadratic $\mathcal{R}$-module $F\{\mathcal{R}\}$. This correspondence yields an equivalence of categories.*

We are especially interested in the case when $\mathcal{R}$ is a ring R (then $\mathcal{Add}(R)$ is the category of finitely generated free R-modules) or when $\mathcal{R}$ is the ringoid $\mathcal{Cyc}$ which is the full subcategory of $\mathcal{Ab}$ consisting of cyclic groups $\mathbb{Z}/p^i$ of prime power order and

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$\mathbb{Z}$ (then $\mathcal{Add}(\mathcal{Cyc})$ is the category of finitely generated abelian groups). But also the topological ringoid consisting of elementary Moore spaces $M(\mathbb{Z}, n) = S^n$ and $M(\mathbb{Z}/p^i, n)$ is important for the computation of homotopy groups of Moore spaces, see Remark 9.2.

In case the ringoid $\mathcal{R}$ is the ring $\mathbb{Z}$ of integers, a quadratic $\mathbb{Z}$-module is the same as a Q-module where Q is the ring described by generators and relations in Proposition 2.2. The quadratic functor $\mathcal{Add}(\mathbb{Z}) \to \mathcal{Ab}$, corresponding to Q by Theorem 3.6, is the direct sum of the tensor square $\otimes^2$ and the quadratic construction P^2 in Definition 2.9.

For the proof of Theorem 3.6 we use the *quadratic tensor product* $A \otimes_{\mathcal{R}} M \in \mathcal{Ab}$ where A is an $\mathcal{R}^{\mathrm{op}}$-module and where M is a quadratic $\mathcal{R}$-module, we also introduce the *quadratic* Hom-*functor* for which $\mathrm{Hom}_{\mathcal{R}}(B, M) \in \mathcal{Ab}$, see Sections 4 and 5. For quadratic functors $F: \mathcal{Ab} \to \mathcal{Ab}$ and $G: \mathcal{Ab}^{\mathrm{op}} \to \mathcal{Ab}$ one has the quadratic approximations (4.2) and (5.2),

$$\lambda: A \otimes_{\mathbb{Z}} F\{\mathbb{Z}\} \to F(A), \qquad \lambda': G(A) \to \mathrm{Hom}_{\mathbb{Z}}(A, G\{\mathbb{Z}\}),$$

which are natural in $A \in \mathcal{Ab}$. Here $F\{\mathbb{Z}\}$ and $G\{\mathbb{Z}\}$ are quadratic $\mathbb{Z}$-modules corresponding to F and G respectively. For the classical functors

$$F = \otimes^2, P^2, \Lambda^2, S^2, \Gamma$$

the quadratic approximation λ is an isomorphism, see Lemma 2.11 and Proposition 4.5. We introduce *derived functors* of the quadratic tensor product $\otimes$ and the quadratic Hom-functor respectively in Section 7 and in Appendices A and B. They only partially coincide with the derived functors in the sense of Dold and Puppe [15].

We need such quadratic derived functors of $\otimes$ and Hom for new natural six-term exact sequences in homotopy theory. The sequences are useful for the computation of the homotopy groups $\pi_m M(A, n)$ of a Moore space and the homology $H_m K(A, n)$ and the cohomology $H^m K(A, n)$ of an Eilenberg–Mac Lane space in the metastable range. In particular the naturality of these exact sequences yields insight in the functorial properties of these groups. We now describe the exact sequence for $\pi_m M(A, n)$; the sequences for $H_m K(A, n)$ and $H^m K(A, n)$ are of a similar nature, see (10.3) and Theorem 10.3.

Theorem 9.3. *For $m < 3n - 2$ there is a natural exact sequence ($A \in \mathcal{Ab}$)*

$$0 \to A *' \pi_m\{S^n\} \to {}_{\lambda}\pi_{m+1} M(A, n) \to A *'' \pi_{m-1}\{S^n\}$$

$$\xrightarrow{\partial} A \otimes \pi_m\{S^n\} \to \pi_m M(A, n) \to {}_{\lambda}\pi_m M(A, n) \to 0.$$

Here ${}_\lambda\pi_m M(A, n)$ *is the cokernel of* $i_*: \pi_m M(A, n)^n \to \pi_m M(A, n)$ *where* i *is the inclusion of the n-skeleton* $M(A, n)^n$. *Moreover,* $\pi_m\{S^n\}$ *is the quadratic* $\mathbb{Z}$*-module given by homotopy group of spheres*

$$\pi_m\{S^n\} = \left(\pi_m(S^n) \xrightarrow{H} \pi_m(S^{2n-1}) \xrightarrow{P} \pi_m(S^n)\right).$$

The map H *is the Hopf invariant and* $P = [i_n, i_n]_*$ *is induced by the Whitehead square. The operators* $*'$ *and* $*''$ *are derived from the quadratic tensor product, see* (7.4).

Various examples of explicit computations of $\pi_m M(A, n)$ are given at the end of Section 9. Using the exact sequence in the theorem we obtain in Example 9.8 a new homotopy invariant

$$\tau(M) \in H_n(M) *' \pi_{2n-1}\{S^n\}$$

of an $(n-1)$-connected $(2n+1)$-dimensional closed manifold M, or more generally Poincaré complex M. The torsion invariant $\tau(M)$ is an analogue of the invariant

$$\varepsilon(N) \in H_n(N) \otimes \pi_{2n-1}\{S^n\}$$

which determines the homotopy type of an $(n-1)$-connected $(2n)$-dimensional Poincaré complex N and which essentially was used by Kervaire and Milnor [20], see Example 9.7. In [9] we describe the connection of $\varepsilon(N)$ with the α-invariant [35] of Wall which classifies $(n-1)$-connected $(2n)$-dimensional manifolds.

For the curious functors R and Ω of Eilenberg and MacLane [16] with $H_5K(A, 2) \cong R(A)$ and $H_7K(A, 3) \cong \Omega(A) \oplus (A \otimes \mathbb{Z}/3)$ we get a new interpretation by the natural isomorphism (see Remark 10.7 and Example 10.4)

$$RA \cong A *' \mathbb{Z}^\Gamma, \quad \text{and} \quad \Omega A \cong A \otimes' \mathbb{Z}^\Gamma.$$

Here $\mathbb{Z}^\Gamma = \pi_3\{S^2\}$ is the quadratic $\mathbb{Z}$-module $\mathbb{Z}^\Gamma = (\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z})$ for which $\Gamma(A) = A \otimes \mathbb{Z}^\Gamma$ is Whitehead's quadratic functor [37]. Also $\otimes'$ is derived from the quadratic tensor product, see (7.4).

Further significant applications of the new quadratic algebra discussed in this paper are described in [7, Chapter II, Section 7] and in [9]. We also use results of this paper in a crucial way for the classification of 2-connected 6-dimensional homotopy types.

1. Modules

We fix some basic notations on categories, ringoids, rings and modules respectively, compare also [24]. A script letter like $\mathscr{C}$ denotes a category, $\mathrm{Ob}(\mathscr{C})$ and $\mathrm{Mor}(\mathscr{C})$ are the classes of objects and morphisms respectively. We identify an object A with its identity $1_A = 1 = A$. We also write $f \in \mathscr{C}$ if f is a morphism or an object in $\mathscr{C}$. The set of morphisms $A \to B$ is $\mathscr{C}(A, B)$. Surjective maps and injective maps are indicated by arrows $\twoheadrightarrow$ and $\rightarrowtail$ respectively.

A *ringoid* $\mathscr{R}$ is a category for which all morphism sets are abelian groups and for which composition is bilinear (equivalently a ringoid is a category enriched over the monoidal category of abelian groups). A ringoid is called a "pre additive category", or an $\mathscr{A}b$-category, see [22]. We prefer the notion "ringoid" since in this paper a ringoid will play the role of a ring. In fact, a ringoid $\mathscr{R}$ with a single object e will be identified with the ring R given by the morphism set $R = \mathscr{R}(e, e)$. Recall that a *biproduct* (or a direct sum) in a ringoid $\mathscr{R}$ is a diagram

$$X \underset{r_1}{\overset{i_1}{\rightleftarrows}} X \vee Y \underset{r_2}{\overset{i_2}{\leftrightarrows}} Y \tag{1.1}$$

which satisfies $r_1 i_1 = 1$, $r_2 i_2 = 1$ and $i_1 r_1 + i_2 r_2 = 1$. Sums and products in a ringoid are as well biproducts, see [22]. An *additive category* is a ringoid in which biproducts exist. Clearly the category $\mathscr{A}b$ of abelian groups is an additive category with biproducts denoted by $X \oplus Y$. A functor $F: \mathscr{R} \to \mathscr{S}$ between ringoids is *additive* if

$$F(f+g) = F(f) + F(g) \tag{1.2}$$

for morphisms $f, g \in \mathscr{R}(X, Y)$. Moreover, we say that F is *quadratic* if Δ, with

$$\Delta(f, g) = F(f+g) - F(f) - F(g), \tag{1.3}$$

is a bilinear function. A module with coefficients in a ringoid $\mathscr{R}$ or equivalently an *$\mathscr{R}$-module* is an additive functor

$$M: \mathscr{R} \to \mathscr{A}b. \tag{1.4}$$

In case $\mathscr{R}$ has only one object e we identify $M = M(e)$ with a module over a ring in the usual sense. An $\mathscr{R}$-module is also called a *left* $\mathscr{R}$-module. A *right* $\mathscr{R}$-module N is a contravariant additive functor $N: \mathscr{R} \to \mathscr{A}b$. For $f \in \mathscr{R}(X, Y)$ we use the notation

$$\begin{aligned} M(f)(x) &= f_*(x) = f \cdot x \quad \text{for } x \in M(X), \\ N(f)(y) &= f^*(y) = y \cdot f \quad \text{for } y \in N(Y). \end{aligned} \tag{1.5}$$

A right $\mathscr{R}$-module is the same as an $\mathscr{R}^{\mathrm{op}}$-module where $\mathscr{R}^{\mathrm{op}}$ is the opposite category. In case $\mathscr{R}$ is small (that is, if the class of objects in $\mathscr{R}$ is a set) let $\mathscr{M}(\mathscr{R})$ be *the category of $\mathscr{R}$-modules*. Morphisms in $\mathscr{M}(\mathscr{R})$ are natural transformations. The category $\mathscr{M}(\mathscr{R})$

is an abelian category; as an example one has $\mathcal{M}(\mathbb{Z}) = \mathcal{Ab}$. We now recall the definition of tensor products of modules.

Definition 1.1. Let $\mathcal{R}$ be a small ringoid, let A be an $\mathcal{R}^{\mathrm{op}}$-module and let B be an $\mathcal{R}$-module. The *tensor product* $A \otimes_{\mathcal{R}} B$ is the abelian group generated by the elements $a \otimes b$, $a \in A(X)$, $b \in B(X)$ where X is any object in $\mathcal{R}$. The relations are

$$(a + a') \otimes b = a \otimes b + a' \otimes b,$$

$$a \otimes (b + b') = a \otimes b + a \otimes b',$$

$$(a'' \cdot \varphi) \otimes b = a'' \otimes (\varphi \cdot b),$$

for $a, a' \in A(X)$, $b, b' \in B(X)$, $\varphi: X \to Y \in \mathcal{R}$, $a'' \in A(Y)$. The tensor product is a biadditive functor $\otimes_{\mathcal{R}}: \mathcal{M}(\mathcal{R}^{\mathrm{op}}) \times \mathcal{M}(\mathcal{R}) \to \mathcal{Ab}$.

Definition 1.2. The *tensor product* $\mathcal{R} \otimes \mathcal{S}$ *of ringoids* $\mathcal{R}, \mathcal{S}$ is the following ringoid. Objects are pairs (X, Y) with $X \in \mathrm{Ob}(\mathcal{R})$, $Y \in \mathrm{Ob}(\mathcal{S})$ and the morphisms $(X, Y) \to (X', Y')$ are the elements of the tensor product of abelian groups $\mathcal{R}(X, X') \otimes_{\mathbb{Z}} \mathcal{S}(Y, Y')$. Composition is defined by $(f \otimes g)(f' \otimes g') = (ff') \otimes (gg')$. Any *biadditive* functor $F: \mathcal{R} \times \mathcal{S} \to \mathcal{Ab}$ has a unique additive factorization (as well denoted by F) $F: \mathcal{R} \otimes \mathcal{S} \to \mathcal{Ab}$ with $F(f \otimes g) = F(f, g)$. For example an $\mathcal{R}$-module A and an $\mathcal{S}$-module $\mathcal{B}$ yield the $\mathcal{R} \otimes \mathcal{S}$-module $A \otimes B$ given by $(A \otimes B)(f \otimes g) = A(f) \otimes_{\mathbb{Z}} B(g)$.

2. Quadratic $\mathbb{Z}$-modules

Let $\mathcal{Add}(\mathbb{Z})$ be the category of finitely generated free abelian groups. The additive functors $F: \mathcal{Add}(\mathbb{Z}) \to \mathcal{Ab}$ are in one–one correspondence with abelian groups. The correspondence is given by $F \mapsto F(\mathbb{Z})$. In this section we introduce quadratic $\mathbb{Z}$-modules which are in one–one correspondence with quadratic functors $\mathcal{Add}(\mathbb{Z}) \to \mathcal{Ab}$. In this sense a quadratic $\mathbb{Z}$-module is just the "quadratic analogue" of an abelian group.

Definition 2.1. A *quadratic $\mathbb{Z}$-module*

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

is a pair of abelian groups M_e, M_{ee} together with homomorphisms H, P which satisfy $PHP = 2P$ and $HPH = 2H$. A morphism $f: M \to N$ between quadratic $\mathbb{Z}$-modules is a pair of homomorphisms $f: M_e \to N_e$, $f: M_{ee} \to N_{ee}$ which commute with H and P respectively. Let $\mathcal{QM}(\mathbb{Z})$ be the category of quadratic $\mathbb{Z}$-modules. For a quadratic

$\mathbb{Z}$-module M we define the *involution* $T = HP - 1 : M_{ee} \to M_{ee}$. Then the equations for H and P are equivalent to $PT = P$ and $TH = H$. Moreover, we get $TT = 1$ since $1 + T = HP = HPT = T + T^2$. We define for $n \in \mathbb{Z}$ the function

$$n_* : M_e \to M_e, \qquad n_*(x) = nx + (n(n-1)/2)PH(x), \quad x \in M_e.$$

One can check that $(n \cdot m)_* = n_* m_*$ and that $(n + m)_* = n_* + m_* + nmPH$. Let $\mathbb{Z}/n = \mathbb{Z}/n\mathbb{Z}$, $n \geq 0$, be the cyclic group of order n. We call M a *quadratic* $\mathbb{Z}/n$*-module* if $n \cdot M_{ee} = 0$ and $n_* M_e = 0$.

We identify a quadratic $\mathbb{Z}$-module M satisfying $M_{ee} = 0$ with the abelian group M_e, this yields the full inclusion $\mathscr{Ab} = \mathscr{M}(\mathbb{Z}) \subset \mathscr{QM}(\mathbb{Z})$. Next we observe that there is a *duality* functor $D : \mathscr{QM}(\mathbb{Z}) \to \mathscr{QM}(\mathbb{Z})$ with $D(M)$ given by the interchange of the roles of H and P respectively, that is,

$$D(M) = \left((DM)_e \xrightarrow{H^D} (DM)_{ee} \xrightarrow{P^D} (DM)_e \right)$$

with $(DM)_e = M_{ee}$ and $(DM)_{ee} = M_e$ and $H^D = P$ and $P^D = H$. Clearly $DD(M) = M$. Moreover, an additive functor $A : \mathscr{Ab} \to \mathscr{Ab}$ induces a functor $A : \mathscr{QM}(\mathbb{Z}) \to \mathscr{QM}(\mathbb{Z})$. Here we define the quadratic $\mathbb{Z}$-module $A(M)$ by $A(M)_e = A(M_e)$ and $A(M)_{ee} = A(M_{ee})$ with H and P given by $A(H)$ and $A(P)$ respectively. For example the functor $A = \otimes_{\mathbb{Z}} C$, $C \in \mathscr{Ab}$, carries M to $[M] \otimes_{\mathbb{Z}} C$.

Proposition 2.2. *There is a ring Q together with an isomorphism $\chi : \mathscr{QM}(\mathbb{Z}) \cong \mathscr{M}(Q)$ of categories where $\mathscr{M}(Q)$ is the category of Q-modules.*

Proof. For $M \in \mathscr{QM}(\mathbb{Z})$ we have inclusions and projections ($\tau = e, ee$)

$$(1) \qquad M_\tau \xrightarrow{i_\tau} M_e \oplus M_{ee} \xrightarrow{r_\tau} M_\tau$$

They yield the following endomorphisms of the abelian group $M_e \oplus M_{ee}$,

$$(2) \qquad a = i_e r_e, \quad b = i_{ee} r_{ee}, \quad h = i_{ee} H r_e, \quad p = i_e P r_{ee},$$

which satisfy the relations

$$(3) \qquad a^2 = a, \quad b^2 = b, \quad ab = ba = 0, \quad a + b = 1,$$
$$ah = 0, \quad hb = 0, \quad pa = 0, \quad bp = 0, \quad php = 2p, \quad hph = 2h.$$

Let Q be the ring generated by a, b, h, p such that the relations are satisfied. Then χ in Proposition 2.2 carries M to the Q-module $M_e \oplus M_{ee}$ defined by (2). As a $\mathbb{Z}$-module

Q is given by $Q = \mathbb{Z}^6$ with basis (a, b, h, p, ph, hp). Moreover, the quadratic $\mathbb{Z}$-module $\chi^{-1}(Q)$, as well denoted by Q, is given by

(4) $\quad Q_e = a \cdot Q = \mathbb{Z}^3$ with basis (a, ap, aph),

$\quad Q_{ee} = b \cdot Q = \mathbb{Z}^3$ with basis (b, bh, bhp),

and by

$$H = P = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad T = HP - 1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

The ring Q was obtained in a more general context by Pirashvili [26]. In fact, Pirashvili defines a ring $Q(n)$ for which the category of $Q(n)$-modules is isomorphic to the category of polynomial functors F from $\mathcal{Add}(\mathbb{Z})$ to $\mathcal{Ab}$ of degree n with $F(0) = 0$. He does not give a description of $Q(2) = Q$ as in (3) above. Recently W. Dreckmann computed for small n the following rank of the free abelian group $Q(n)$, this rank is

1, 6, 39, 320, 3281, 40558, 586751,

9719616, 181353777, 3762893750,

85934344775, 2141853777856, 57852105131809,

1683237633305502, 52483648929669119

for $n = 1, \ldots, 15$. Many results on quadratic $\mathbb{Z}$-modules in this paper should have generalizations for $Q(n)$-modules. $\square$

Recall that an object X in an additive category is *indecomposable* if X admits no isomorphism $X \cong A \oplus B$ with $A \neq 0$ and $B \neq 0$. It is an interesting problem to classify all finitely generated indecomposable quadratic $\mathbb{Z}$-modules up to isomorphism. This leads to the following examples. We say that a quadratic $\mathbb{Z}$-module M is of *cyclic type* if M_e and M_{ee} are cyclic groups. Let $1_n \in \mathbb{Z}/n$ be the generator and let $k : \mathbb{Z}/n \to \mathbb{Z}/m$ be the homomorphism with $k(1_n) = k \cdot 1_m$, $k \in \mathbb{Z}$, $m \mid k \cdot n$. Then we obtain the list in Table 1 where $C = \mathbb{Z}$ or $C = \mathbb{Z}/p^i$, $p =$ prime, $s, t \geq 1$.

The isomorphic objects in the list are given by $C^\Gamma \cong C^S$ if $C = \mathbb{Z}/q^i$ (q *odd*). With the notations in Definition 2.1 we clearly have $C^\Gamma = [\mathbb{Z}^\Gamma] \otimes_{\mathbb{Z}} C$, $C^S = [\mathbb{Z}^S] \otimes_{\mathbb{Z}} C$ and $C^\Lambda = [\mathbb{Z}^\Lambda] \otimes_{\mathbb{Z}} C$. We leave it to the reader to describe the dualities in the list. An elementary but somewhat elaborate proof shows the following:

Proposition 2.3. *The quadratic $\mathbb{Z}$-modules in Table 1 furnish a complete list of indecomposable quadratic $\mathbb{Z}$-modules of cyclic type.* $\square$

Table 1

M	M_e	M_{ee}	H	P
C	C	0	0	0
C^{Λ}	0	C	0	0
C^{Γ}	C	C	1	2
C^{S}	C	C	2	1
$H(t)$	$\mathbb{Z}$	$\mathbb{Z}/2^t$	2^{t-1}	0
$P(s)$	$\mathbb{Z}/2^s$	$\mathbb{Z}$	0	2^{s-1}
$s+t>1$, $H(s,t)$	$\mathbb{Z}/2^s$	$\mathbb{Z}/2^t$	2^{t-1}	0
$s+t>1$, $P(s,t)$	$\mathbb{Z}/2^s$	$\mathbb{Z}/2^t$	0	2^{s-1}
$s+t>1$, $M(s+t)$	$\mathbb{Z}/2^s$	$\mathbb{Z}/2^t$	2^{t-1}	2^{s-1}
$\Gamma(s)$	$\mathbb{Z}/2^{s+1}$	$\mathbb{Z}/2^s$	1	2
$S(s)$	$\mathbb{Z}/2^s$	$\mathbb{Z}/2^{s+1}$	2	1
$s>1$, $\Gamma'(s)$	$\mathbb{Z}/2^{s+1}$	$\mathbb{Z}/2^s$	$2^{s-1}+1$	2
$s>1$, $S'(s)$	$\mathbb{Z}/2^s$	$\mathbb{Z}/2^{s+1}$	2	$2^{s-1}+1$

Remark. It would be interesting to know a complete list of all indecomposable quadratic $\mathbb{Z}$-modules. However, to furnish such a list is an intricate problem of representation theory. It might be helpful to consider the more general problem of finding indecomposable Λ-representations of the quiver

$$\bullet \leftrightarrows \bullet$$

(cf. for example [38, Vol. II, Section 77]). Indeed, if $\Lambda = \mathbb{Z}[\mathbb{Z}/2]$ is the group ring of the cyclic group $\mathbb{Z}/2$, then a quadratic $\mathbb{Z}$-module is a representation of this quiver given by Λ-homomorphisms $H: M_e \to M_{ee}$, $p: M_{ee} \to M_e$ where Λ acts via T on M_{ee} and acts trivially on M_e. Here one can use the fact that the indecomposable $\mathbb{Z}[\mathbb{Z}/2]$-lattices are known, see [38, Vol. I, (34.31)]. Such lattices are part of quadratic $\mathbb{Z}$-modules M for which M_e and M_{ee} are finitely generated free $\mathbb{Z}$-modules like for example $\mathbb{Z}^{\otimes}$ and $\mathbb{Z}^P$ in Remark 2.8; compare [38].

Definition 2.4. Let $F: \mathscr{R} \to \mathscr{Ab}$ be a quadratic functor and let $X \vee Y$ be a biproduct in $\mathscr{R}$. The *quadratic cross effect* $F(X \mid Y)$ is defined by the image group

(1) $$F(X \mid Y) = \operatorname{im}\{\Delta(i_1 r_1, i_2 r_2): F(X \vee Y) \to F(X \vee Y)\}$$

see (1.3) and (1.1). If $\mathscr{R}$ is an additive category we get by (1) the biadditive functor

(2) $$F(-|-): \mathscr{R} \times \mathscr{R} \to \mathscr{Ab}.$$

Moreover, we have the isomorphism

(3) $$\Psi: F(X) \oplus F(Y) \oplus F(X \mid Y) \simeq F(X \vee Y)$$

which is given by $F(i_1)$, $F(i_2)$ and the *inclusion* $i_{12}: F(X \mid Y) \subset F(X \vee Y)$. Let r_{12} be the *retraction* of i_{12} obtained by Ψ^{-1} and by projection to $F(X \mid Y)$. For the biproduct $X \vee X$ one has the maps $\mu = i_1 + i_2 : X \to X \vee X$ and $\nabla = r_1 + r_2 : X \vee X \to X$. They yield homomorphisms H and P with

$$(4) \qquad F\{X\} = \left(F(X) \xrightarrow{H} F(X \mid X) \xrightarrow{P} F(X) \right)$$

by $H = r_{12}F(\mu)$ and $P = F(\nabla)i_{12}$. Moreover, we derive from $f + g = \nabla(f \vee g)\mu$ the formula

$$(5) \qquad F(f+g) = F(f) + F(g) + PF(f \mid g)H$$

or equivalently $\Delta(f, g) = PF(f \mid g)H$, see (1.3).

Proposition 2.5. *Let $F: \mathcal{R} \to \mathcal{Ab}$ be a quadratic functor and assume $\mathcal{R}$ is an additive category. Then $F\{X\}$ is a quadratic $\mathbb{Z}$-module and $X \mapsto F\{X\}$ defines a functor $\mathcal{R} \to \mathcal{QM}(\mathbb{Z})$.*

Proof. We define the interchange map

$$(1) \qquad t: X \vee X \to X \vee X, \qquad t = i_2 r_1 + i_1 r_2$$

Then we have $t\mu = \mu$ and $\nabla t = \nabla$. Moreover, t induces a map

$$(2) \qquad T: F(X \mid X) \to F(X \mid X)$$

with $F(t)i_{12} = i_{12}T$ and $r_{12}F(t) = Tr_{12}$. Hence we get $TH = H$ and $PT = P$. Moreover, we obtain $HP = 1 + T$ by applying F to the following commutative diagram in $\mathcal{R}$:

$$(3) \qquad \begin{array}{ccccc} X \vee X & \xrightarrow{\nabla} & X & \xrightarrow{\mu} & X \vee X \\ \downarrow{\scriptstyle \mu \vee \mu} & & & & \downarrow{\scriptstyle \nabla \vee \nabla} \\ X \vee X \vee X \vee X & & \xrightarrow{1 \vee t \vee 1} & & X \vee X \vee X \vee X \end{array}$$

Here we use the biadditivity of $F(-\mid-)$ in Definition 2.4(2). □

The significance of quadratic $\mathbb{Z}$-modules is described by the next result which is a special case of Theorem 3.6 below. Let $\mathcal{Add}(\mathbb{Z}/n)$ be the full subcategory of $\mathcal{Ab}$ consisting of finitely generated free $(\mathbb{Z}/n)$-modules, $n \geq 0$ (for $n = 0$ we set $\mathbb{Z}/0 = \mathbb{Z}$).

Theorem 2.6. *There is a 1–1 correspondence between quadratic functors $F: \mathcal{Add}(\mathbb{Z}/n) \to \mathcal{Ab}$ and quadratic $\mathbb{Z}/n$-modules M, $n \geq 0$. The correspondence carries F to $F\{\mathbb{Z}/n\}$, see Definition* 2.4(4). $\square$

Here a "1–1 correspondence" denotes a bijection which maps isomorphism classes to isomorphism classes. Hence any quadratic functor $F: \mathcal{Add}(\mathbb{Z}/n) \to \mathcal{Ab}$ is completely determined (up to isomorphism) by the fairly simple algebraic data of the quadratic $\mathbb{Z}$-module $F\{\mathbb{Z}/n\}$ which is actually a quadratic $\mathbb{Z}/n$-module. In addition to the correspondence in Theorem 2.6 we obtain in Theorem 3.6 below an equivalence of categories.

The next result shows that the universal quadratic $\mathbb{Z}$-module Q in (2.1) is actually decomposable.

Proposition 2.7. *One has an isomorphism $Q \cong \mathbb{Z}^P \oplus \mathbb{Z}^{\otimes}$ of quadratic $\mathbb{Z}$-modules where*

$$\mathbb{Z}^{\otimes} = (\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}) \quad \textit{and} \quad \mathbb{Z}^P = (\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z})$$

are given by $H = (1,1)$ and $P = (1,1)$. Here $\mathbb{Z}^P$ is the dual of $\mathbb{Z}^{\otimes}$, that is $\mathbb{Z}^P = D\mathbb{Z}^{\otimes}$.

Proof. The isomorphism is given by the matrices

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ for } Q_e \quad \text{and} \quad \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ for } Q_{ee}. \qquad \square$$

Remark 2.8. The quadratic $\mathbb{Z}$-modules $\mathbb{Z}^{\otimes}$ and $\mathbb{Z}^P$ are unique in the following sense. Up to isomorphism there is only one indecomposable quadratic $\mathbb{Z}$-module M with $M_e = \mathbb{Z}$ and $M_{ee} = \mathbb{Z} \oplus \mathbb{Z}$, namely $M \cong \mathbb{Z}^{\otimes}$. Dually there is up to isomorphism only one indecomposable quadratic $\mathbb{Z}$-module N with $N_e = \mathbb{Z} \oplus \mathbb{Z}$ and $N_{ee} = \mathbb{Z}$, namely $N \cong \mathbb{Z}^P$. For example $\mathbb{Z}^P$ is isomorphic to the following two quadratic $\mathbb{Z}$-modules

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(1,0)} \mathbb{Z} \xrightarrow{(2,-1)} \mathbb{Z} \oplus \mathbb{Z}, \qquad \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(2,1)} \mathbb{Z} \xrightarrow{(1,0)} \mathbb{Z} \oplus \mathbb{Z}.$$

The quadratic $\mathbb{Z}$-modules $\mathbb{Z}^{\otimes}$ and $\mathbb{Z}^P$ correspond to classical quadratic functors $\otimes^2$ and P^2 which we define as follows.

Definition 2.9. The *tensor square* $\otimes^2$ is the quadratic functor

$$\otimes^2: \mathcal{Ab} \to \mathcal{Ab} \quad \text{with } \otimes^2(A) = A \otimes_{\mathbb{Z}} A.$$

The *quadratic construction* P^2 is the functor

$$P^2: \mathcal{Ab} \to \mathcal{Ab} \quad \text{with } P^2(A) = \Delta(A)/\Delta^3(A).$$

Here $\Delta(A)$ is the augmentation ideal in the group ring $\mathbb{Z}[A]$ and $\Delta^3(A)$ is the third power.

Remark 2.10. A function $f: A \to B$ between abelian groups is *weak quadratic* if

(1) $\qquad [a, b]_f = f(a + b) - f(a) - f(b)$

is bilinear for $a,b \in A$. Moreover, f is *quadratic* if in addition $f(-a) = f(a)$. The function

(2) $\qquad \tilde{\gamma}: A \to P^2(A),$

which carries $a \in A$ to the element represented by $|a| - 1 \in \Delta(A)$, is the *universal weak quadratic function*. That is, each weak quadratic function f admits a unique factorization $f = f^{\square}\tilde{\gamma}$ where $f^{\square}: P^2(A) \to B$ is a homomorphism. Whitehead's quadratic functor $\Gamma: \mathcal{Ab} \to \mathcal{Ab}$ is defined by the *universal quadratic function*

(3) $\qquad \gamma: A \to \Gamma(A),$

see [37]. Using the functor Γ the functor P^2 can also be described by the following natural pull-back diagram in $\mathcal{Ab}$ which has short exact rows ($A \in \mathcal{Ab}$):

(4)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^2(A) & \xrightarrow{\tilde{\omega}} & P^2(A) & \xrightarrow{1^{\square}} & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \gamma^{\square}} & & \downarrow{\scriptstyle q} & & \\ 0 & \longrightarrow & S^2(A) & \xrightarrow{\omega} & \Gamma(A) & \xrightarrow{q^{\square}} & A \otimes \mathbb{Z}/2 & \longrightarrow & 0 \end{array}$$

Here q is the quotient map which is a quadratic function so that $q^{\square}$ is defined. Moreover, the *symmetric square* S^2 is the functor

(5) $\qquad S^2: \mathcal{Ab} \to \mathcal{Ab}, \qquad S^2(A) = A \otimes A/\{a \otimes b - b \otimes a \sim 0\}.$

The map $\tilde{\omega}$ in (4) is defined by $\tilde{\omega}\{a \otimes b\} = \tilde{\gamma}(a + b) - \tilde{\gamma}(a) - \tilde{\gamma}(b)$, see (1). We also shall use the *exterior square*

(6) $\qquad \Lambda^2: \mathcal{Ab} \to \mathcal{Ab}, \qquad \Lambda^2(A) = A \otimes A/\{a \otimes a \sim 0\},$

which is part of a natural exact sequence

(7) $\qquad \Gamma(A) \xrightarrow{H} \otimes^2(A) \xrightarrow{q} \Lambda^2(A) \to 0$

where $H = h^{\square}$ is defined by $h(a) = a \otimes a$ and where q is the quotient map. Using Definition 2.4(4) we obtain for each quadratic functor $F: \mathcal{Ab} \to \mathcal{Ab}$ the quadratic $\mathbb{Z}$-module $F\{\mathbb{Z}\}$. As special cases we now obtain the following lemma (see Proposition 2.7 and Definition 2.4).

Lemma 2.11. *One has isomorphisms of quadratic $\mathbb{Z}$-modules*

$$\otimes^2\{\mathbb{Z}\} \cong \mathbb{Z}^{\otimes} = D\mathbb{Z}^P, \quad P^2\{\mathbb{Z}\} \cong \mathbb{Z}^P = D\mathbb{Z}^{\otimes}, \quad \Lambda^2\{\mathbb{Z}\} \cong \mathbb{Z}^{\Lambda} = D\mathbb{Z},$$
$$\Gamma\{\mathbb{Z}\} \cong \mathbb{Z}^{\Gamma} = D\mathbb{Z}^S, \qquad S^2\{\mathbb{Z}\} \cong \mathbb{Z}^S = D\mathbb{Z}^{\Gamma}.$$

The basis of $\otimes^2(\mathbb{Z} \mid \mathbb{Z}) \cong (\mathbb{Z}^{\otimes})_{ee} = \mathbb{Z} \oplus \mathbb{Z}$ is $(e_1 \otimes e_2, e_2 \otimes e_1)$ where (e_1, e_2) is the canonical basis of $\mathbb{Z} \oplus \mathbb{Z}$. Moreover, the basis of $P^2(\mathbb{Z}) \cong (\mathbb{Z}^P)_e = \mathbb{Z} \oplus \mathbb{Z}$ is $(\tilde{\gamma}(1), \tilde{\omega}(1 \otimes 1) - \tilde{\gamma}(1))$, where we use $\tilde{\gamma}$ and $\tilde{\omega}$ of Remark 2.10.

Various results in this section are proved carefully in the Diplomarbeit of my student V. Jeschonnek [19]. This Diplomarbeit contains also further interesting results on the homological algebra of Q-modules where Q is the ring in Proposition 2.2.

3. Quadratic R-modules

We consider quadratic $\mathcal{R}$-modules where $\mathcal{R}$ is ringoid. For $\mathcal{R} = \mathbb{Z}$ they are just the quadratic $\mathbb{Z}$-modules discussed in Section 2.

Definition 3.1. Let $\mathcal{R}$ be a ringoid. A *quadratic $\mathcal{R}$-module* $M = (M_e, M_{ee}, T, H, P)$ is a pair of functors $M_e: \mathcal{R} \to \mathcal{Ab}$, $M_{ee}: \mathcal{R} \times \mathcal{R} \to \mathcal{Ab}$ (both as well denoted by M) together with natural transformations

$$T = T_{X,Y}: M(X, Y) \to M(Y, X), \quad M(X) \xrightarrow{H} M(X, X) \xrightarrow{P} M(X)$$

such that the following properties are satisfied:

(1) $PT = P$,
(2) $TH = H$,
(3) $T = HP - 1$ on $M(X, X)$,
(4) $TT = 1$.

Moreover, the functor M_{ee} is biadditive and the functor M_e is quadratic with

$$(5) \qquad M(f + g) = M(f) + M(g) + PM(f, g)H$$

for $f, g: X \to Y \in \mathcal{R}$. We also write $f_* = M(f)$ and $(f, g)_* = M(f, g)$. A morphism $F: M \to N$ between quadratic $\mathcal{R}$-modules is a pair of natural transformations

$$(6) \qquad F_e: M_e \to N_e, \qquad F_{ee}: M_{ee} \to N_{ee}$$

which commute with T, H, and P respectively. Let $\mathcal{QM}(\mathcal{R})$ be the *category of quadratic $\mathcal{R}$-modules* for a small ringoid $\mathcal{R}$.

We identify a quadratic $\mathcal{R}$-module, satisfying $M_{ee} = 0$, with an $\mathcal{R}$-module. This yields the full inclusion of abelian categories $\mathcal{M}(\mathcal{R}) \subset \mathcal{QM}(\mathcal{R})$, see (2.1). On the other hand a quadratic $\mathcal{R}$-module M with $M_e = 0$ is the same as a pair (M_{ee}, T) where M_{ee} a biadditive functor $\mathcal{R} \times \mathcal{R} \to \mathcal{Ab}$ and where $T = T_{X,Y}: M_{ee}(X, Y) \cong M_{ee}(Y, X)$ is a natural transformation with $TT = 1$ and $T_{X,X} = -1$, $X, Y \in \mathrm{Ob}\, \mathcal{R}$. The direct sum $M \oplus N$ of quadratic $\mathcal{R}$-modules is given by $(M \oplus N)_e(X) = M_e(X) \oplus N_e(X)$ and $(M \oplus N)_{ee}(X, Y) = M_{ee}(X, Y) \oplus N_{ee}(X, Y)$.

Remark 3.2. For the ringoid $\mathcal{R} = \mathbb{Z}$ a quadratic $\mathcal{R}$-module M is the same as a quadratic $\mathbb{Z}$-module with $M_e = M(e)$, $M_{ee} = M(e, e)$. In fact, for $n \in \mathcal{R}(e, e) = \mathbb{Z}$ the induced map $M(n) = n_*$ is defined in Definition 2.1 and $T = T_{e,e}$ in Definition 3.1 is defined by T in Definition 2.1. This also shows that for the ring $\mathcal{R} = \mathbb{Z}/n$ a quadratic $\mathcal{R}$-module is the same as a quadratic $\mathbb{Z}/n$-module defined in Definition 2.1.

The equations (1), (2) and (3) in Definition 3.1 for a quadratic $\mathcal{R}$-module show that for $X \in \mathrm{Ob}(\mathcal{R})$

$$M\{X\} = \left(M(X) \xrightarrow{H} M(X, X) \xrightarrow{P} M(X) \right) \tag{3.1}$$

is a quadratic $\mathbb{Z}$-module. Hence M yields a functor $M: \mathcal{R} \to \mathcal{QM}(\mathbb{Z})$ which carries the object X to $M\{X\}$. The quadratic $\mathcal{R}$-module M, however, is not determined by this functor since for example $T_{X,Y}$ in Definition 3.1 is given for all pairs $(X, Y) \in \mathrm{Ob}(\mathcal{R}) \times \mathrm{Ob}(\mathcal{R})$. In case $\mathcal{R}$ has a single object e, that is, if $\mathcal{R} = R$ is a ring, then a *quadratic R-module M* consists of quadratic $\mathbb{Z}$-module

(1) $\quad M(e) \xrightarrow{H} M(e, e) \xrightarrow{P} M(e).$

Here $M(e, e)$ is an $R \otimes_{\mathbb{Z}} R$-module and the multiplicative monoid of R acts on $M(e)$ such that H and P are equivariant with respect to the diagonal action on $M(e, e)$ and such that

(2) $\quad (f + g)_*(x) = f_*(x) + g_*(x) + P((f \otimes g) \cdot (Hx)).$

Here $f_*(x)$ denotes the action of $f \in R$ on $x \in M(e)$.

Examples 3.3. Let R be a *commutative* ring. We define quadratic R-modules R^A, R^S, and R^Γ as follows:

M	$M(e)$	$M(e,e)$	H	P
R^A	0	R	0	0
R^S	R	R	2	1
R^Γ	R	R	1	2

Here $f \in R$ acts on $x \in M(e)$ by $f_*(x) = f \cdot f \cdot x$ and $f \otimes g$ acts on $y \in M(e,e)$ by $(f \otimes g) \cdot y = f \cdot g \cdot y$.

Example 3.4. Let $\mathscr{R}$ be a ringoid, let $\mathscr{A}$ be an additive category, and let $i: \mathscr{R} \to \mathscr{A}$ be an additive functor. Often $\mathscr{R}$ is a subringoid of $\mathscr{A}$ and i is the inclusion, for example $\mathscr{R} = \mathscr{A}$. Then any quadratic functor $F: \mathscr{A} \to \mathscr{Ab}$ yields a quadratic $\mathscr{R}$-module

$$F\{\mathscr{R}\} = i^*F = (F_e, F_{ee}, T, H, P)$$

as follows. The functors $F_e = i^*F$ and $F_{ee} = (i \times i)^* F(-|-)$ are the restrictions of the functors F and $F(-|-)$, see Definition 2.4. Moreover, H, P and T are given as in Definition 2.4 and in the proof of Proposition 2.5, respectively. In case $\mathscr{R}$ is the subringoid generated by the identity $1_X \in \mathrm{Ob}(\mathscr{A})$, $F\{\mathscr{R}\}$ is the same as the quadratic $\mathbb{Z}$-module $F\{X\}$ in Proposition 2.5.

We now are ready to describe the generalization of Theorem 2.6 for quadratic $\mathscr{R}$-modules; for this we recall from [22, Chapter VIII, Section 2] the following definition:

Definition 3.5. Let $\mathscr{R}$ be a ringoid. Then the free additive category

(1) $\qquad i: \mathscr{R} \subset \mathscr{Add}(\mathscr{R})$

is given as follows. The objects of $\mathscr{Add}(\mathscr{R})$ are the n-tuples $X = (X_1, \ldots, X_n)$ of objects X_i in $\mathscr{R}$, $0 \leq n < \infty$. The morphisms are the corresponding matrices of morphisms in $\mathscr{R}$. The inclusion i carries the object X to the corresponding tuple of length 1. Any additive functor $f: \mathscr{R} \to \mathscr{A}$ (where $\mathscr{A}$ is an additive category) has a unique additive extension $\bar{f}: \mathscr{Add}(\mathscr{R}) \to \mathscr{A}$ which carries the tuple X to the n-fold biproduct $\bar{f}(X) = fX_1 \vee \cdots \vee fX_n$ in $\mathscr{A}$. Let $\mathscr{Quad}(\mathscr{R})$ be the category of quadratic functors

(2) $\qquad F: \mathscr{Add}(\mathscr{R}) \to \mathscr{Ab}$,

morphisms are natural transformations.

Theorem 3.6. *There is an equivalence of categories* $\mathcal{Q}uad(\mathcal{R}) \simeq \mathcal{QM}(\mathcal{R})$ *which carries* F *to the restriction* $F\{\mathcal{R}\}$ *in Example* 3.4. $\square$

For a ring $\mathcal{R} = R$ the category $\mathcal{A}dd(R)$ coincides with the full subcategory of finitely generated free R-modules in $\mathcal{M}(R)$. Therefore, Theorem 2.6 is readily obtained by Theorem 3.6. The inverse of the equivalence in Theorem 3.6 is given by the tensor products defined in the next section; one obtains Theorem 3.6 as a corollary of Proposition 4.3.

4. The quadratic tensor product

We introduce the tensor product of an $\mathcal{R}^{\mathrm{op}}$-module and a quadratic $\mathcal{R}$-module. This is the quadratic generalization of the tensor product defined in Definition 1.1.

Definition 4.1. Let $\mathcal{R}$ be a small ringoid. We define the functor

$$\otimes_{\mathcal{R}} : \mathcal{M}(\mathcal{R}^{\mathrm{op}}) \times \mathcal{QM}(\mathcal{R}) \to \mathcal{A}b$$

which carries the pair (A, M) to the *tensor product* $A \otimes_{\mathcal{R}} M$. The abelian group $A \otimes_{\mathcal{R}} M$ is generated by the symbols

$$(1) \qquad \begin{array}{ll} a \otimes m, & a \in A(X),\ m \in M(X), \\ [a, b] \otimes n, & a \in A(X),\ b \in A(Y),\ n \in M(X, Y), \end{array}$$

where X, Y are objects in $\mathcal{R}$. The relations are

$$(2) \qquad \begin{array}{l} (a + b) \otimes m = a \otimes m + b \otimes m + [a, b] \otimes H(m), \\ a \otimes (m + m') = a \otimes m + a \otimes m', \\ [a, a] \otimes n = a \otimes P(n), \\ [a, b] \otimes n = [b, a] \otimes T(n), \\ [a, b] \otimes n \text{ is linear in each variable } a, b, \text{ and } n, \\ (\varphi^* a) \otimes m = a \otimes (\varphi_* m), \\ [\varphi^* a, \Psi^* b] \otimes n = [a, b] \otimes (\varphi, \Psi)_*(n), \end{array}$$

where φ, Ψ are morphisms in $\mathcal{R}$ and where a, b, m, m', n are appropriate elements as in (1). (We point out that the last two equations of (2) are redundant if $\mathcal{R} = \mathbb{Z}$.) For morphisms $F: A \to A' \in \mathcal{M}(\mathcal{R}^{\mathrm{op}})$ and $G: M \to M' \in \mathcal{Q}\mathcal{M}(\mathcal{R})$ we define the induced homomorphism

(3) $$F \otimes G : A \otimes_{\mathcal{R}} M \to A' \otimes_{\mathcal{R}} M'$$

by the formulas

(4) $$(F \otimes G)(a \otimes m) = (Fa) \otimes (G_e m),$$
$$(F \otimes G)([a, b] \otimes n) = [Fa, Fb] \otimes (G_{ee} n).$$

In case $M_{ee} = 0$ we see that $A \otimes_{\mathcal{R}} M$ coincides with the tensor product of Definition 1.1.

Proposition 4.2. *The tensor product of Definition* 4.1 *yields an additive functor*

(1) $$A \otimes_{\mathcal{R}} (-) : \mathcal{Q}\mathcal{M}(\mathcal{R}) \to \mathcal{A}b$$

for each A in $\mathcal{M}(\mathcal{R})$ *and a quadratic functor*

(2) $$(-) \otimes_{\mathcal{R}} M : \mathcal{M}(\mathcal{R}^{\mathrm{op}}) \to \mathcal{A}b$$

for each M in $\mathcal{Q}\mathcal{M}(\mathcal{R})$. *The quadratic cross effect of* (2) *is given by the formula*

(3) $$(A \mid B) \otimes_{\mathcal{R}} M = (A \otimes B) \otimes_{\mathcal{R} \otimes \mathcal{R}} M_{ee}. \qquad \square$$

Here A and B are $\mathcal{R}^{\mathrm{op}}$-modules which yield the $(\mathcal{R} \otimes \mathcal{R})^{\mathrm{op}}$-module $A \otimes B$ by Definition 1.2 and the $\mathcal{R} \otimes \mathcal{R}$-module M_{ee} is given by M. The right-hand side of (3) is a tensor product in the sense of Definition 1.1. The isomorphism (3) is obtained by the inclusion

(4) $$i_{12} : (A \otimes B) \otimes_{\mathcal{R} \otimes \mathcal{R}} M_{ee} \rightarrowtail (A \oplus B) \otimes_{\mathcal{R}} M$$

which carries $a \otimes b \otimes n$ to $[i_1 a, i_2 b] \otimes n$ for $a \in A(X)$, $b \in B(Y)$, $n \in M(X, Y)$. By Example 3.4 the quadratic functor $F = (-) \otimes_{\mathcal{R}} M$ is as well a quadratic $\mathcal{M}(\mathcal{R})$-module. Here the structure maps T, H, P are given by the natural transformations

(5) $$(A \otimes B) \otimes_{\mathcal{R} \otimes \mathcal{R}} M_{ee} \xrightarrow{T} (B \otimes A) \otimes_{\mathcal{R} \otimes \mathcal{R}} M_{ee},$$

(6) $$A \otimes_{\mathcal{R}} M \xrightarrow{H} (A \otimes A) \otimes_{\mathcal{R} \otimes \mathcal{R}} M_{ee} \xrightarrow{P} A \otimes_{\mathcal{R}} M,$$

defined by the formulas

$$
(7)\qquad
\begin{aligned}
&H(a\otimes m) = (a\otimes a)\otimes H(m),\\
&H([a,b]\otimes n) = (a\otimes b)\otimes n + (b\otimes a)\otimes T(n),\\
&T((a\otimes b)\otimes n) = (b\otimes a)\otimes T(n),\\
&P((a\otimes b)\otimes n) = [a,b]\otimes n.
\end{aligned}
$$

We point out that the tensor product of Definition 4.1 is compatible with direct limits in $\mathcal{M}(\mathcal{R}^{\mathrm{op}})$ and $\mathcal{Q}\mathcal{M}(\mathcal{R})$ respectively.

Let $\mathcal{A}$ be an additive category and let $F:\mathcal{A}\to\mathcal{A}b$ be a quadratic functor. For a small subringoid $\mathcal{R}\subset\mathcal{A}$ the quadratic $\mathcal{R}$-module $F\{\mathcal{R}\}$ is defined by Example 3.4. On the other hand each object U in $\mathcal{A}$ gives us the $\mathcal{R}^{\mathrm{op}}$-module

$$[\mathcal{R},U]:\mathcal{R}^{\mathrm{op}}\to\mathcal{A}b$$

which carries $X\in\mathcal{R}$ to $\mathcal{A}(X,U)=[X,U]$. We now define a map

$$\lambda:[\mathcal{R},U]\otimes_{\mathcal{R}}F\{\mathcal{R}\}\to F(U) \tag{4.1}$$

by $\lambda(a\otimes m)=F(a)(m)$ for $a\in[X,U]$, $m\in F(X)$ and $\lambda([a,b]\otimes n)=PF(a\,|\,b)(n)$ for $b\in[Y,U]$ and $n\in F(X\,|\,Y)$.

Proposition 4.3. *The homomorphism λ in* (4.1) *is well defined and natural. Moreover, λ is an isomorphism if $U=X_1\vee\cdots\vee X_r$ is a finite biproduct with $X_i\in\mathcal{R}$ for $i=1,\ldots,r$ and if $\mathcal{R}$ is a full subringoid of $\mathcal{A}$.* □

This is a crucial property of the tensor product of Definition 4.1 which shows that this definition is naturally derived from the notion of a quadratic functor. The proposition shows that a quadratic functor $F:\mathcal{A}dd(\mathcal{R})\to\mathcal{A}b$ is completely determined by the quadratic $\mathcal{R}$-module $F\{\mathcal{R}\}=i^*F$. This proves Theorem 3.6; in fact, the inverse of the functor of Theorem 3.6 carries $M\in\mathcal{Q}\mathcal{M}(\mathcal{R})$ to the quadratic functor $[\mathcal{R},-]\otimes_{\mathcal{R}}M$.

The next corollary illustrates Proposition 4.3. Let $\mathcal{C}yc$ be the full subcategory of $\mathcal{A}b$ consisting of cyclic groups $\mathbb{Z}/n$ where $n=0$ or where n is a prime power. Then we have the equivalence of categories

$$\mathcal{A}dd(\mathcal{C}yc)\xrightarrow{\sim}\mathcal{F}\mathcal{A}b \tag{4.2}$$

where $\mathcal{F}\mathcal{A}b$ is the full category of $\mathcal{A}b$ consisting of finitely generated abelian groups. Since each abelian group is the limit of its finitely generated subgroups we get the following corollary:

Corollary 4.4. *Let $F : \mathcal{Ab} \to \mathcal{Ab}$ be a quadratic functor which commutes with direct limits. Then F is completely determined by the quadratic $\mathcal{Cyc}$-module $F\{\mathcal{Cyc}\}$, see Example 3.4. In fact, we have the natural isomorphism $[\mathcal{Cyc}, A] \otimes_{\mathcal{Cyc}} F\{\mathcal{Cyc}\} \simeq F(A)$ for A in $\mathcal{Ab}$.* $\square$

We now consider examples of the natural transformation λ in (4.1). A *commutative* ring R satisfies $R^{\mathrm{op}} = R$. Therefore, we get for any quadratic functor $F : \mathcal{M}(R) \to \mathcal{Ab}$ the natural homomorphism ($A \in \mathrm{Ob}\,\mathcal{M}(R)$)

$$\lambda : A \otimes_R F\{R\} \to F(A). \tag{4.3}$$

Here the quadratic R-module $F\{R\}$ is essentially given by the homomorphisms in $\mathcal{Ab}$

$$F(R) \xrightarrow{H} F(R\,|\,R) \xrightarrow{P} F(R),$$

see Definition 2.4(4) and (1) below (3.1), and λ is defined as follows. For $a \in A$ let $\bar{a} : R \to A$ be the map in $\mathcal{M}(R)$ with $\bar{a}(1) = a$. Then we get for $m \in F(R)$ and $n \in F(R\,|\,R)$ the formulas $\lambda(a \otimes m) = F(\bar{a})(m)$ and $\lambda([a,b] \otimes n) = PF(\bar{a}\,|\,\bar{b})(n)$. By Proposition 4.3 the map λ is an isomorphism if A is a finitely generated free R-module. We call λ the *tensor approximation* of the quadratic functor F. For $R = \mathbb{Z}$ we have the following examples for which the tensor approximation is actually a natural isomorphism.

Proposition 4.5. *The quadratic functors $F = \otimes^2, P^2, \Lambda^2, \Gamma, S^2$ in Remark 2.10 satisfy $A \otimes_{\mathbb{Z}} F\{\mathbb{Z}\} \cong F(A)$ for $A \in \mathcal{Ab}$, hence one has natural isomorphisms*

$$\otimes^2(A) \cong A \otimes \mathbb{Z}^{\otimes}, \quad P^2(A) \cong A \otimes \mathbb{Z}^P, \quad \Lambda^2(A) \cong A \otimes \mathbb{Z}^{\Lambda},$$

$$\Gamma(A) \cong A \otimes \mathbb{Z}^{\Gamma}, \qquad S^2(A) \cong A \otimes \mathbb{Z}^S. \qquad \square$$

The torsion functor $F : \mathcal{Ab} \to \mathcal{Ab}$ with $F(A) = A * A$, however, is a quadratic functor for which the tensor approximation is no isomorphism, in fact, $F\{\mathbb{Z}\} = 0$ in this case. One can check Proposition 4.8 by the definition of the relations in Definition 4.1. Finally we observe the next result where we use the notation $[M] \otimes_{\mathbb{Z}} C$ in Definition 2.1.

Proposition 4.6. *For $M \in \mathcal{QM}(\mathbb{Z})$ and $A, C \in \mathcal{Ab}$ we have the natural isomorphism*

$$A \otimes_{\mathbb{Z}} ([M] \otimes_{\mathbb{Z}} C) \simeq (A \otimes_{\mathbb{Z}} M) \otimes_{\mathbb{Z}} C. \qquad \square$$

5. The quadratic Hom functor

Let $\mathscr{R}$ be a small ringoid. For $\mathscr{R}$-modules A, B one has the abelian group $\mathrm{Hom}_{\mathscr{R}}(A, B)$ which consists of all natural transformations $A \to B$. We now extend this Hom functor for the case that B is a quadratic $\mathscr{R}$-module.

Definitions 5.1. We define the functor

$$\mathrm{Hom}_{\mathscr{R}} : \mathscr{M}(\mathscr{R})^{\mathrm{op}} \times \mathscr{Q}\mathscr{M}(\mathscr{R}) \to \mathscr{Ab}$$

which carries the pair (A, M) to the abelian group $\mathrm{Hom}_{\mathscr{R}}(A, M)$, the elements of which are called *quadratic forms* $A \to M$ *over* $\mathscr{R}$. A quadratic form $\alpha : A \to M$ is given by functions $(X, Y \in \mathrm{Ob}(\mathscr{R}))$

$$(1) \qquad \alpha_X : A(X) \to M(X), \qquad \alpha_{X,Y} : A(X) \times A(Y) \to M(X, Y)$$

such that the following properties are satisfied; (they are analogous to the corresponding properties in Definition 4.1(2) and they as well define the sum $\alpha + \beta$ of quadratic forms).

$$(2) \qquad \begin{aligned} &\alpha_X(a + b) = \alpha_X(a) + \alpha_X(b) + P\alpha_{X,X}(a, b), \\ &(\alpha + \beta)_X = \alpha_X + \beta_X, \\ &\alpha_{X,X}(a, a) = H\alpha_X(a), \\ &\alpha_{X,Y}(a, b) = T\alpha_{Y,X}(b, a), \\ &\alpha_{X,Y} \text{ is bilinear and } (\alpha + \beta)_{X,Y} = \alpha_{X,Y} + \beta_{X,Y}, \\ &M_e(\varphi)\alpha_X = \alpha_{X_1} A(\varphi), \\ &M_{ee}(\varphi, \Psi)\alpha_{X,Y} = \alpha_{X_1,Y_1}(A(\varphi) \times A(\Psi)). \end{aligned}$$

Here a, b are appropriate elements in $A(X)$ or $A(Y)$ and $\varphi : X \to X_1$, $\Psi : Y \to Y_1$ are morphisms in $\mathscr{R}$. The last two equations describe the "naturality" of the quadratic form α (these equations are redundant if $\mathscr{R} = \mathbb{Z}$). For morphisms $F : A' \to A$ in $\mathscr{M}(\mathscr{R})$ and $G : M \to M' \in \mathscr{Q}\mathscr{M}(\mathscr{R})$ we define the induced homomorphisms

$$(3) \qquad \mathrm{Hom}(F, G) : \mathrm{Hom}_{\mathscr{R}}(A, M) \to \mathrm{Hom}_{\mathscr{R}}(A', M')$$

by the formulas $\mathrm{Hom}(F, G)(\alpha) = \beta$ with

$$(4) \qquad \beta_X = G_e \alpha_X F, \qquad \beta_{X,Y} = G_{ee} \alpha_{X,Y}(F \times F).$$

In case $M_{ee} = 0$ we see that $\mathrm{Hom}_{\mathscr{R}}(A, M)$ coincides with the usual group of natural transformations $A \to M$, hence the functor of Definition 5.1 extends canonically the classical functor $\mathrm{Hom}_{\mathscr{R}}$ for $\mathscr{R}$-modules.

Proposition 5.2. *The* Hom-*functor of Definition* 5.1 *yields an additive functor*

(1) $$\mathrm{Hom}_{\mathscr{R}}(A, -): \mathscr{Q}\mathscr{M}(\mathscr{R}) \to \mathscr{A}b$$

for each A in $\mathscr{M}(\mathscr{R})$ and a quadratic functor

(2) $$\mathrm{Hom}_{\mathscr{R}}(-, M): \mathscr{M}(\mathscr{R})^{\mathrm{op}} \to \mathscr{A}b$$

for each M in $\mathscr{Q}\mathscr{M}(\mathscr{R})$. The quadratic cross effect of (2) *is given by the formula*

(3) $$\mathrm{Hom}_{\mathscr{R}}(A \mid B, M) = \mathrm{Hom}_{\mathscr{R} \otimes \mathscr{R}}(A \otimes B, M_{ee}). \qquad \square$$

Compare Proposition 4.2 where we describe the corresponding result for quadratic tensor products. The isomorphism in (3) is obtained by the projection

(4) $$r_{12}: \mathrm{Hom}_{\mathscr{R}}(A \oplus B, M) \twoheadrightarrow \mathrm{Hom}_{\mathscr{R} \otimes \mathscr{R}}(A \otimes B, M_{ee})$$

which carries α to the natural transformation $\beta: A(X) \otimes B(Y) \to M_{ee}(X, Y)$ with $\beta(a \otimes b) = \alpha_{X,Y}(i_1 a, i_2 b)$. By Example 3.4 the quadratic functor $F = \mathrm{Hom}_{\mathscr{R}}(-, M)$ is a quadratic $\mathscr{M}(\mathscr{R})^{\mathrm{op}}$-module; the structure maps T, H, P are given by the following natural transformations

(5) $$\mathrm{Hom}_{\mathscr{R} \otimes \mathscr{R}}(A \otimes B, M_{ee}) \xrightarrow{T} \mathrm{Hom}_{\mathscr{R} \otimes \mathscr{R}}(B \otimes A, M_{ee}),$$

(6) $$\mathrm{Hom}_{\mathscr{R}}(A, M) \xrightarrow{H} \mathrm{Hom}_{\mathscr{R} \otimes \mathscr{R}}(A \otimes B, M_{ee}) \xrightarrow{P} \mathrm{Hom}_{\mathscr{R}}(A, M),$$

defined by

(7) $$\begin{aligned} &(T\beta)(a \otimes b) = T\beta(b \otimes a), \\ &(H\alpha)(a \otimes b) = \alpha(a, b) + T\alpha(b, a), \\ &(P\beta)(a) = H\beta(a \otimes a) \text{ and } (P\beta)(a, b) = \beta(a \otimes b). \end{aligned}$$

Examples 5.3. Let R be a commutative ring and consider the quadratic R-modules R^{A}, R^{S} and R^{Γ} defined in Example 3.3. Moreover, let A be an R-module.

(1) A quadratic form $\alpha: A \to R^{\Lambda}$ can be identified with an R-bilinear map $\alpha: A \times A \to R$ satisfying $\alpha(a, a) = 0$. Hence α is just an alternating *bilinear form.*

(2) A quadratic form $\alpha: A \to R^{S}$ can be identified with a function $\alpha: A \to R$ which satisfies $\alpha(\lambda \cdot a) = \lambda^2 \cdot \alpha(a)$ for $\lambda \in R$, $a \in A$ and for which the function

$$\Delta_\alpha: A \times A \to R, \qquad \Delta_\alpha(a, b) = \alpha(a + b) - \alpha(a) - \alpha(b)$$

is R-bilinear. Thus α is the same as a *quadratic form on* A in the classical sense, see for example [1, 29].

(3) A quadratic form $\alpha: A \to R^{\Gamma}$ can be identified with a pair of functions $\alpha: A \to R$, $\Delta: A \times A \to R$ for which $\alpha(\lambda a) = \lambda^2 \alpha(a)$ and for which Δ is symmetric R-bilinear with $2\Delta(a, b) = \alpha(a + b) - \alpha(a) - \alpha(b)$ and $\Delta(a, a) = \alpha(a)$. If R is uniquely 2-divisible α is a special quadratic form as in (2) since in this case Δ is determined by α.

Lemma 5.4. *Let R be a ring and let F be a finitely generated free R-module. Then $\mathrm{Hom}_R(F, R)$ is an R^{op}-module such that*

$$\chi: \mathrm{Hom}_R(F, R) \otimes_R M \cong \mathrm{Hom}_R(F, M)$$

for any quadratic R-module M.

Proof. We define the natural isomorphism χ as follows. Let $a, b \in \mathrm{Hom}_R(F, R)$, $m \in M(e)$, $n \in M(e, e)$. Then $\chi(a \otimes m) = \alpha$ is given by $\alpha(x) = M_e(a(x))(m)$ and $\alpha(x, y) = M_{ee}(a(x), a(y))H(m)$ for $x, y \in F$. Moreover, $\chi([a, b] \otimes n) = \beta$ is given by $\beta(x) = PM_{ee}(a(x), b(x))(n)$ and $\beta(x, y) = M_{ee}(a(x), b(y))(n) + M_{ee}(a(y), b(x))(n)$. □

Let $\mathscr{A}$ be an additive category and let $F: \mathscr{A}^{\mathrm{op}} \to \mathscr{Ab}$ be a quadratic functor. For a small subringoid $\mathscr{R} \subset \mathscr{A}$ the quadratic $\mathscr{R}^{\mathrm{op}}$-module $F\{\mathscr{R}^{\mathrm{op}}\}$ is defined as in Example 3.4 by $\mathscr{R}^{\mathrm{op}} \subset \mathscr{A}^{\mathrm{op}}$. On the other hand each object U in $\mathscr{A}$ gives the $\mathscr{R}^{\mathrm{op}}$-module $[\mathscr{R}, U]$ as in (4.1). We now define the map

$$\lambda: F(U) \to \mathrm{Hom}_{\mathscr{R}^{\mathrm{op}}}([\mathscr{R}, U], F\{\mathscr{R}^{\mathrm{op}}\}) \tag{5.1}$$

as follows. For $\xi \in F(U)$ let $\lambda(\xi)$ be given by the functions $\alpha_X, \alpha_{X,Y}$ $(X, Y \in \mathscr{R}^{\mathrm{op}})$ with $\alpha_X(a) = a^*(\xi) = F(a)(\xi)$, $a \in [X, U]$ and $\alpha_{X,Y}(a, b) = F(a \mid b)H(\xi)$, $b \in [Y, U]$.

Proposition 5.5. *The homomorphism λ is an isomorphism if $U = X_1 \vee \cdots \vee X_r$ is a finite biproduct with $X_i \in \mathscr{R}$ for $i = 1, \ldots, r$ and if $\mathscr{R}$ is a full subringoid of $\mathscr{A}$.* □

This result is a crucial property of the Hom-group in Definition 5.1 which shows that this definition is again naturally derived from the notion of a quadratic functor. We leave it to the reader to formulate a corollary of Proposition 5.5 corresponding to

Corollary 4.4. Moreover, we get as in (4.3) the following example. Let R be a commutative ring and let $F:\mathcal{M}(R)^{\mathrm{op}} \to \mathcal{Ab}$ be a quadratic functor. Then the quadratic R-module $F\{R\}$ is defined and we derive from (5.1) the natural transformation

$$\lambda: F(A) \to \mathrm{Hom}_R(A, F\{R\}) \tag{5.2}$$

where $A \in \mathcal{M}(R)$, compare (4.3). By Proposition 5.5 this map is an isomorphism if A is a finitely generated free R-module. We call (5.2) the Hom-*approximation* of the quadratic functor F.

6. The quadratic chain functors

In this section we associate with each quadratic $\mathcal{R}$-module M quadratic chain functors M_* and M^*. The definition of M_* and M^* is motivated by the applications in homotopy theory below. The quadratic chain functors as well form a first step for the construction of derived functors.

Let $\mathcal{R}$ be a ringoid with a zero object denoted by 0. A *chain complex* $X_* = (X_*, d)$ in $\mathcal{R}$ is a sequence of maps in $\mathcal{R}$

$$\cdots \to X_n \overset{d}{\to} X_{n-1} \overset{d}{\to} \cdots \quad (n \in \mathbb{Z}) \tag{6.1}$$

with $dd = 0$. A chain map $F: X_* \to Y_*$ is given by maps $F = F_n: X_n \to Y_n$ with $dF = Fd$ and a chain homotopy $\alpha: F \simeq G$ is given by maps $\alpha = \alpha_n: X_{n-1} \to Y_n$ with $-F_n + G_n = \alpha_n d + d\alpha_{n+1}$. The chain complex X_* is *positive* (*negative*) if $X_i = 0$ for $i < 0$ ($X_i = 0$ for $i > 0$). A negative chain complex is also called a *cochain complex* X^* where we write $X^n = X_{-n}$, $d: X^n \to X^{n+1}$. Let $\mathcal{R}_*$ ($\mathcal{R}^*$) be the category of positive (negative) chain complexes and let $\mathcal{R}_*/\simeq$ ($\mathcal{R}^*/\simeq$) be its homotopy category.

We also need the category $\mathcal{P}air(\mathcal{R})$ of *pairs in* $\mathcal{R}$; objects are morphisms d in $\mathcal{R}$ and maps $F: d \to d'$, $F = (F_A, F_B)$, are commutative diagrams

$$\begin{tikzcd} A \arrow[r, "F_A"] \arrow[d, "d"'] & A' \arrow[d, "d'"] \\ B \arrow[r, "F_B"] & B' \end{tikzcd} \tag{6.2}$$

A homotopy $\alpha: F \simeq G$ is a map $\alpha: B \to A'$ with $-F_A + G_A = \alpha d$, $-F_B + G_B = d'\alpha$. We have full inclusions of $\mathcal{P}air(\mathcal{R})/\simeq$ into $R_*/\simeq$ and $R^*/\simeq$ which carry d to the chain complex $d: A = X_1 \to B = X_0$ and to the cochain complex $d: A = X^0 \to B = X^1$ respectively.

Definition 6.1. Let M be a quadratic $\mathscr{R}$-module. The *quadratic chain functors* associated to M are functors

(1) $\qquad M_*:\mathscr{P}air(\mathscr{R})\to \mathscr{A}b_*, \qquad M^*:\mathscr{P}air(\mathscr{R})\to \mathscr{A}b^*$

which are defined as follows. For an object $d:X_1\to X_0$ in $\mathscr{P}air(\mathscr{R})$ we define the chain complex $M_*(d)$ by $M_i(d)=0$ for $i>2$ and by

(2)
$$\begin{array}{ccccc} M(X_1,X_1) & \xrightarrow{(P,\,-(1,d)_*)} & M(X_1)\oplus M(X_1,X_0) & \xrightarrow{(d_*,\,P(d,1)_*)} & M(X_0) \\ \| & & \| & & \| \\ M_2(d) & & M_1(d) & & M_0(d) \end{array}$$

On the other hand we define for an object $d:X^0\to X^1$ in $\mathscr{P}air(\mathscr{R})$ the cochain complex $M^*(d)$ by $M^i(d)=0$ for $i>2$ and by

(3)
$$\begin{array}{ccccc} M(X^0) & \xrightarrow{(d_*,\,(d,1)_*H)} & M(X^1)\oplus M(X^1,X^0) & \xrightarrow{(H,\,-(1,d)_*)} & M(X^1,X^1) \\ \| & & \| & & \| \\ M^0(d) & & M^1(d) & & M^2(d) \end{array}$$

For a map $F:d\to d''$ in $\mathscr{P}air(\mathscr{R})$ the induced chain maps $M_*(F)$ and $M^*(F)$ are defined in the obvious way. One readily checks that the composition of maps in (2) and (3) respectively is the trivial map 0. The definition of M_*, M^* is motivated by the examples in [8].

We point out that the definition of M^* above is dual to the definition of M_*; here duality is obtained by reversing arrows and by the interchange of H and P.

Theorem 6.2. *The quadratic chain functors of Definition* 6.1 *induce functors*

$$M_*:\mathscr{P}air(\mathscr{R})/\simeq\ \to \mathscr{A}b_*/\simeq, \qquad M^*:\mathscr{P}air(\mathscr{R})/\simeq\ \to \mathscr{A}b^*/\simeq$$

between homotopy categories.

Proof. Let $f=(f_1,f_0)$ and $g=(g_1,g_0)$ be maps $d\to d'$ in $\mathscr{P}air(\mathscr{R})$ and let $\alpha:f\simeq g$ be a homotopy. We can define a homotopy

(1) $\qquad \beta:M_*(f)\simeq M_*(g)$

by the matrices (2) and (3),

(2) $\qquad \beta_1=\begin{pmatrix} B_1\\ B_2\end{pmatrix}$ with $B_1=\alpha_*$, $B_2=(\alpha,f_0)_*H$,

(3) $\qquad \beta_2=(A_1,A_2)$ with $A_1=(\alpha d,f_1)_*H$, $A_2=-(g_1,\alpha)_*+T(f_1,\alpha)_*$.

For the proof of (1) we have to check the following equations (4)–(9):

$$-f_{0*} + g_{0*} = d_* B_1 + P(d, 1)_* B_2, \tag{4}$$

$$-(f_1, f_1)_* + (g_1, g_1)_* = A_1 P - A_2(1, d)_*, \tag{5}$$

$$-f_{1*} + g_{1*} = PA_1 + B_1 d_*, \tag{6}$$

$$-(f_1, f_0)_* + (g_1, g_0)_* = -(1, d)_* A_2 + B_2 P(d, 1)_*, \tag{7}$$

$$0 = PA_2 + B_1 P(d, 1)_*, \tag{8}$$

$$0 = -(1, d)_* A_1 + B_2 d_*. \tag{9}$$

Originally we found the formulas in (2) and (3) as a solution of the system of equations (4)–(9). We now check (4).

$$\begin{aligned} & d_* \alpha_* + P(d, 1)_* (d, f_0)_* H \\ &= (d\alpha)_* + P(d\alpha, f_0)_* H = (d\alpha)_* + (d\alpha + f_0)_0 - (d\alpha)_* - f_{0*} \\ &= g_{0*} - f_{0*}. \end{aligned} \tag{10}$$

Here we use Definition 3.1(5) and $d\alpha = -f_0 + g_0$. Next we obtain (5) by $\alpha d = -f_1 + g_1$ and by Definition 3.1(3):

$$\begin{aligned} & (\alpha d, f_1)_* HP + (g_1, \alpha)_* (1, d)_* - T(f_1, \alpha)_* (1, d)_* \\ &= (\alpha d, f_1)_* T + (\alpha d, f_1)_* + (g_1, \alpha d)_* - T(f_1, \alpha d)_* \\ &= (\alpha d, f_1)_* + (g_1, \alpha d)_* = (-f_1 + g_1, f_1)_* + (g_1, -f_1 + g_1)_* \\ &= -(f_1, f_1)_* + (g_1, g_1)_*. \end{aligned} \tag{11}$$

In the last equation we use the biadditivity of the functor M_{ee} in Definition 3.1. For equation (6) we consider

$$P(\alpha d, f_1)_* H + \alpha_* d_* = (\alpha d + f_1)_* - (\alpha d)_* - f_{1*} + (\alpha d)_* = -f_{1*} + g_{1*}. \tag{12}$$

Next, equation (7) follows from

$$\begin{aligned} & -(1, d)_* (g_1, \alpha)_* - (1, d)_* T(f_1, \alpha)_* + (\alpha, f_0)_* HP(d, 1)_* \\ &= (g_1, d\alpha)_* - (\alpha, df_1)_* T + (\alpha d, f_0)_* + (\alpha, f_0 d)_* T \\ &= (g_1, -f_0 + g_0)_* + (-f_1 + g_1, f_0)_* \\ &= (g_1, g_0)_* - (f_1, f_0)_*. \end{aligned} \tag{13}$$

Moreover, we obtain (8) by

$$(14)\qquad \begin{aligned} &-P(g_1,\alpha)_* + PT(f_1,\alpha)_* + \alpha_* P(d,1)_* \\ &= -P(g_1,\alpha)_* + P(f_1,\alpha)_* + P(\alpha d,\alpha)_* = 0 \end{aligned}$$

In the last equation we use $\alpha d = -f_1 + g_1$. Finally we obtain (9) by

$$(15)\qquad -(1,d)_*(\alpha d, f_1)_* H + (\alpha, f_0)_* H d_* = -(\alpha d, df_1)_* H + (\alpha d, f_0 d)_* H = 0$$

Here we use $df_1 = f_0 d$. This completes the proof of Theorem 6.2 for M_*.

The proof for M^* uses the "dual" arguments. Let $f = (f^0, f^1)$, $g = (g^0, g^1)$ be maps $d' \to d$ in $\mathcal{P}air(\mathcal{R})$ and let $\alpha: f \simeq g$ be a homotopy. Then we define a homotopy

$$(16)\qquad \beta: M^*(f) \simeq M^*(g)$$

by the matrices (17) and (18),

$$(17)\qquad \beta^0 = (B_1, B_2) \quad \text{with } B_1 = \alpha_*,\ B_2 = P(\alpha, f^0)_*$$

$$(18)\qquad \beta^1 = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad \text{with } A_1 = P(\alpha d', f^1)_*,\ A_2 = -(g^1, \alpha)_* + T(f^1, \alpha)_*.$$

One can check as above that (16) is satisfied. □

Remark 6.3. The Dold–Kan theorem shows that positive chain complexes X_* in an abelian category $\mathcal{A}$ are in 1–1 correspondence with simplicial objects $X_0: \Delta^{\mathrm{op}} \to \mathcal{A}$ in $\mathcal{A}$. Here Δ is the simplicial category. The correspondence is given by the functors K and N with $NK(X_*) \cong X_*$ and $KN(X_0) \cong X_0$, see for example [15, Section 3]. Now let $\mathcal{R}$ be a ringoid and let

$$F: \mathcal{A}dd(\mathcal{R}) \to \mathcal{A}b$$

be a quadratic functor. Then F is determined by the quadratic $\mathcal{R}$-module $M = F\{\mathcal{R}\}$ as in Theorem 3.6. Each positive chain complex X_* in $\mathcal{R}$ determines the simplicial object $K(X_*): \Delta^{\mathrm{op}} \to \mathcal{A}dd(\mathcal{R})$ as above since for the definition of the functor K the category $\mathcal{A}$ needs only to be additive. The functor F yields the simplicial object $FK(X_*): \Delta^{\mathrm{op}} \to \mathcal{A}b$ so that the chain complex $NFK(X_*)$ in $\mathcal{A}b$ is defined. If $X_* = (d: X_1 \to X_0)$ is given by a map d in $\mathcal{R}$ with $X_i = 0$ for $i > 1$ one can show that there is a natural homotopy equivalence of chain complexes in $\mathcal{A}b$,

$$NFK(d: X_1 \to X_0) \simeq M_*(d).$$

Here the right-hand side is defined as in Definition 6.1 with $M = F\{\mathscr{R}\}$. We do, however, not see that the dual complex $M^*(d)$ in Definition 6.1 as well has such a property.

7. Quadratic functors induced by a quadratic $\mathbb{Z}$-module

For a $\mathbb{Z}$-module M one has the functors which carry an abelian group A to the group

$$A \otimes M, \quad A * M, \quad \mathrm{Hom}(A, M) \quad \text{and} \quad \mathrm{Ext}(A, M),$$

respectively. We now introduce for a quadratic $\mathbb{Z}$-module M twelve quadratic functors which generalize these classical functors. Using short free resolutions we obtain functors

$$i : \mathscr{Ab} \to \mathscr{Pair}(\mathscr{Ab})/\simeq \quad \text{and} \quad i^{\mathrm{op}} : \mathscr{Ab}^{\mathrm{op}} \to \mathscr{Pair}(\mathscr{Ab}^{\mathrm{op}})/\simeq \tag{7.1}$$

as follows. For each abelian group A we choose a short exact sequence

$$G \overset{d_A}{\rightarrowtail} F \overset{q}{\twoheadrightarrow} A$$

where G and F are free abelian groups and we set $i(A) = d_A$. For a homomorphism $\varphi : A \to B$ we can choose a map $f : d_A \to d_B$ in $\mathscr{Pair}(\mathscr{Ab})$ which induces φ. The homotopy class $\{f\}$ of f is well defined by φ and we set $i(\varphi) = \{f\}$. The functor i is actually full and faithful. The functor i^{op} is induced by i.

A quadratic $\mathbb{Z}$-module M yields the quadratic functors

$$(-) \otimes_{\mathbb{Z}} M : \mathscr{Ab} \to \mathscr{Ab} \quad \text{and} \quad \mathrm{Hom}(-, M) : \mathscr{Ab}^{\mathrm{op}} \to \mathscr{Ab} \tag{7.2}$$

which as well yield a quadratic $\mathscr{Ab}$-module $\{-\} \otimes_{\mathbb{Z}} M$ and a quadratic $\mathscr{Ab}^{\mathrm{op}}$-module $\mathrm{Hom}\{-, M\}$ respectively; compare Proposition 4.2, (5) and (6) and Proposition 5.2, (5) and (6). We now use Theorem 6.2 and (7.1) for the definition of the *quadratic chain functors*,

$$\begin{aligned}
&(\{-\} \otimes_{\mathbb{Z}} M)_* i : \mathscr{Ab} \to \mathscr{Ab}_*/\simeq, \\
&(\{-\} \otimes_{\mathbb{Z}} M)_* i : \mathscr{Ab} \to \mathscr{Ab}^*/\simeq, \\
&(\mathrm{Hom}\{-, M\})_* i^{\mathrm{op}} : \mathscr{Ab}^{\mathrm{op}} \to \mathscr{Ab}_*/\simeq, \\
&(\mathrm{Hom}\{-, M\})^* i^{\mathrm{op}} : \mathscr{Ab}^{\mathrm{op}} \to \mathscr{Ab}^*/\simeq .
\end{aligned} \tag{7.3}$$

The (co)homology groups of these four quadratic chain functors yield six functors $\mathcal{Ab} \to \mathcal{Ab}$ and six functors $\mathcal{Ab}^{\mathrm{op}} \to \mathcal{Ab}$ which we denote as follows where $d_A = i(A)$ as in (7.1) and where $j = 0, 1$, resp. 2.

$$\begin{aligned}
&H_j(\{-\} \otimes_{\mathbb{Z}} M)_* d_A = A \otimes M,\ A *' M,\ \text{resp. } A *'' M,\\
&H^j(\{-\} \otimes_{\mathbb{Z}} M)^* d_A = A * M,\ A \otimes' M,\ \text{resp. } A \otimes'' M,\\
&H_j(\mathrm{Hom}\{-, M\})_* d_A^{\mathrm{op}} = \mathrm{Ext}(A, M),\ \mathrm{Hom}'(A, M),\ \text{resp. } \mathrm{Hom}''(A, M),\\
&H^j(\mathrm{Hom}\{-, M\})^* d_A^{\mathrm{op}} = \mathrm{Hom}(A, M),\ \mathrm{Ext}'(A, M),\ \text{resp. } \mathrm{Ext}''(A, M).
\end{aligned} \tag{7.4}$$

For the convenience of the reader we now describe explicitly the chain complexes used in (7.4). For this we choose $d = d_A : G \to F$ as in (7.1).

(1) The chain complex $(\{-\} \otimes_{\mathbb{Z}} M)_* d_A$ is given by

$$G \otimes G \otimes M_{ee} \xrightarrow{(P,\ -d_*)} G \otimes_{\mathbb{Z}} M \oplus G \otimes F \otimes M_{ee} \xrightarrow{(d_*,\ Pd_*)} F \otimes_{\mathbb{Z}} M.$$

(2) The cochain complex $(\{-\} \otimes_{\mathbb{Z}} M)^* d_A$ is given by

$$F \otimes F \otimes M_{ee} \xleftarrow{(H,\ -d_*)} F \otimes_{\mathbb{Z}} M \oplus F \otimes G \otimes M_{ee} \xleftarrow{(d_*,\ d_* H)} G \otimes_{\mathbb{Z}} M.$$

(3) The chain complex $(\mathrm{Hom}\{-, M\})_* d_A^{\mathrm{op}}$ is given by

$$\mathrm{Hom}(F \otimes F, M_{ee}) \xrightarrow{(P,\ -d^*)} \mathrm{Hom}_{\mathbb{Z}}(F, M) \oplus \mathrm{Hom}(F \otimes G, M_{ee})$$
$$\xrightarrow{(d^*,\ Pd^*)} \mathrm{Hom}_{\mathbb{Z}}(G, M).$$

(4) The cochain complex $(\mathrm{Hom}\{-, M\})^* d_A^{\mathrm{op}}$ is given by

$$\mathrm{Hom}(G \otimes G, M_{ee}) \xleftarrow{(H,\ -d^*)} \mathrm{Hom}_{\mathbb{Z}}(G, M) \oplus \mathrm{Hom}(G \otimes F, M_{ee})$$
$$\xleftarrow{(d^*,\ d^* H)} \mathrm{Hom}_{\mathbb{Z}}(F, M).$$

Here d_*, d^* denote the maps induced by d and the formulas for H and P are described in Proposition 4.2, (7) and 5.2, (7), respectively. The degree of the group at the right-hand side in each sequence above is 0.

The notation in (7.4) is chosen since there is the following compatibility with classical functors. Assume M is a $\mathbb{Z}$-module, that is $M_{ee}=0$, then one readily verifies that the groups

$$A\otimes M = A\otimes' M, \qquad A*M = A*'M,$$

$$\mathrm{Hom}(A, M) = \mathrm{Hom}'(A, M), \qquad \mathrm{Ext}(A, M) = \mathrm{Ext}'(A, M)$$

are given by the corresponding classical functors for abelian groups. Moreover, all groups $A*''M$, $A\otimes''M$, $\mathrm{Hom}''(A, M)$ and $\mathrm{Ext}''(A, M)$ with $j=2$ in (7.4) are trivial for $M_{ee}=0$.

Remark 7.1. Six of the functors in (7.4) are actually derived functors in the sense of Dold–Puppe [15]. For this let $T=(-)\otimes_{\mathbb{Z}} M$ and $T'=\mathrm{Hom}(-, M)$ be the functors in (7.2). Then the *derived functors* $L_iT: \mathcal{Ab}\to\mathcal{Ab}$ and $R^iT': \mathcal{Ab}^{\mathrm{op}}\to\mathcal{Ab}$ are defined by

$$L_iT(A) = H_iNTK(X_*) \quad\text{and}\quad R^iT'(A) = H^iNT'K(X_*)$$

respectively where $X_* = (d_A: F\to G)$ is given by a presentation of A as in (7.1). Now one can show that one has natural isomorphisms

$$L_iT(A) = A\otimes M,\ A*'M,\ \text{resp. } A*''M,$$

$$R^iT'(A) = \mathrm{Hom}(A, M),\ \mathrm{Ext}'(A, M),\ \text{resp. } \mathrm{Ext}''(A, M)$$

for $i=0$, 1, resp. 2. For $L_iT(A)$ this is a direct consequence of the equivalence in Remark 6.3. Since T and T' are quadratic the derived functors above are trivial for $i>2$.

Proposition 7.2. *One has natural isomorphisms* $A\otimes M = A\otimes_{\mathbb{Z}} M$ *and* $\mathrm{Hom}(A, M) = \mathrm{Hom}_{\mathbb{Z}}(A, M)$ *where the right-hand side is defined by Definition* 4.1 *and Definition* 5.1 *respectively. Compare also Theorem* B.9. □

Proposition 7.3. *All functors in* (7.4) *are additive in M and quadratic in A. The quadratic cross effects are naturally given by*

$$(A\,|\,B)\otimes M = A\otimes B\otimes M_{ee} = (A\,|\,B)\otimes'' M,$$

$$(A\,|\,B)*M = A*B*M_{ee} = (A\,|\,B)*''M,$$

$$\mathrm{Ext}(A\,|\,B, M) = \mathrm{Ext}(A*B, M_{ee}) = \mathrm{Ext}''(A\,|\,B, M),$$

$$\mathrm{Hom}(A\,|\,B, M) = \mathrm{Hom}(A\otimes B, M_{ee}) = \mathrm{Hom}''(A\,|\,B, M),$$

$$(A\,|\,B)*'M = H_1(d_A\otimes d_B, M_{ee}) = (A\,|\,B)\otimes' M,$$

$$\mathrm{Hom}'(A\,|\,B, M) = H^1(d_A\otimes d_B, M_{ee}) = \mathrm{Ext}'(A\,|\,B, M).$$

Here d_A denotes as well the chain complex (X_*, d) with $d = d_A: X_1 = G \to X_0 = F$, $X_i = 0$ for $i \geq 2$. The Künneth formula yields natural exact sequences

(1) $$(A * B) \otimes M_{ee} \rightarrowtail H_1(d_A \otimes d_B, M_{ee}) \twoheadrightarrow (A \otimes B) * M_{ee},$$

(2) $$\mathrm{Ext}(A \otimes B, M_{ee}) \rightarrowtail H^1(d_A \otimes d_B, M_{ee}) \twoheadrightarrow \mathrm{Hom}(A * B, M_{ee}).$$

These sequences are split, the splitting however is not natural. There is a natural isomorphism

(3) $$H_1(d_A \otimes d_B, M_{ee}) = \mathrm{Trip}(A, B, M_{ee}),$$

where the right-hand side is the *triple torsion product* of Mac Lane [21].

Proof of Proposition 7.3. We consider for $N = \{-\} \otimes_{\mathbb{Z}} M$ the functor $N_*: \mathcal{P}air(\mathcal{Ab})/\simeq \to \mathcal{Ab}_*/\simeq$, see (7.3). This functor is quadratic and its quadratic cross effect admits a weak equivalence

$$\Psi: N_*(d_A \,|\, d_B) \xrightarrow{\simeq} d_A \otimes d_B \otimes M_{ee}$$

of chain complexes. For $d_A: X_1 \to X_0$ and $d_B: Y_1 \to Y_0$ and $C_* = N_*(d_A \,|\, d_B)$ we have

$$C_0 = X_0 \otimes Y_0 \otimes M_{ee},$$
$$C_1 = X_1 \otimes Y_1 \otimes M_{ee} \oplus X_1 \otimes Y_0 \otimes M_{ee} \oplus Y_1 \otimes X_0 \otimes M_{ee},$$
$$C_2 = X_1 \otimes Y_1 \otimes M_{ee} \oplus Y_1 \otimes X_1 \otimes M_{ee}.$$

The differential $d_i: C_i \to C_{i-1}$ is given by

$$d_2(x_1 \otimes y_1 \otimes n) = x_1 \otimes y_1 \otimes n - x_1 \otimes d_B y_1 \otimes n,$$
$$d_2(y_1 \otimes x_1 \otimes n) = x_1 \otimes y_1 \otimes Tn - y_1 \otimes d_A x_1 \otimes n,$$
$$d_1(x_1 \otimes y_1 \otimes n) = d_A x_1 \otimes d_B y_1 \otimes n,$$
$$d_1(x_1 \otimes y_0 \otimes n) = d_A x_1 \otimes y_0 \otimes n,$$
$$d_1(y_1 \otimes x_0 \otimes n) = x_0 \otimes d_B y_1 \otimes Tn,$$

where $y_i \in Y_i$, $x_i \in X_i$, $n \in M_{ee}$. The map Ψ is given by the identity in degree 0 and by

$$\Psi_2(x_1 \otimes y_1 \otimes n) = 0,$$
$$\Psi_2(y_1 \otimes x_1 \otimes n) = x_1 \otimes y_1 \otimes Tn,$$
$$\Psi_1(x_1 \otimes y_1 \otimes n) = x_1 \otimes d_B y_1 \otimes n,$$
$$\Psi_1(x_1 \otimes y_0 \otimes n) = x_1 \otimes y_0 \otimes n,$$
$$\Psi_1(y_1 \otimes x_0 \otimes n) = x_0 \otimes y_1 \otimes Tn.$$

Since $H_jN_*(d_A \mid d_B)$ is the cross effect in $H_jN_*(d_A \oplus d_B)$ we obtain $(A \mid B) \otimes M$, $(A \mid B) *' M$ and $(A \mid B) *'' M$ by the weak equivalence Ψ and by the Künneth formulae. In a similar way one obtains the other cross effects. $\square$

Proposition 7.4. *There are natural inclusions and projections of abelian groups*

$$A *'' M \rightarrowtail A * A * M_{ee},$$

$$A \otimes'' M \twoheadleftarrow A \otimes A \otimes M_{ee},$$

$$\mathrm{Hom}''(A, M) \rightarrowtail \mathrm{Hom}(A \otimes A, M_{ee}),$$

$$\mathrm{Ext}''(A, M) \twoheadleftarrow \mathrm{Ext}(A * A, M_{ee}).$$

Proof. We only consider the first inclusion. For this we see by (7.4), (1), that $A *'' M$ is the intersection $(d_* = 1 \otimes d \otimes 1)$

$$\ker(P) \cap \ker(-d_*) \subset G \otimes (A * M_{ee}) \subset G \otimes G \otimes M_{ee},$$

where $\ker(-d_*) = G \otimes (A * M_{ee})$. We have to show $(d \otimes 1 \otimes 1)(A *'' M) = 0$. Then the first inclusion is given. Let $T: G \otimes G \otimes M_{ee} \to G \otimes G \otimes M_{ee}$ be the interchange map with $T(x \otimes y \otimes n) = y \otimes x \otimes Tn$. Since $HP = 1 + T$ we see that T restricted to $\ker(P)$ is -1. Whence we get for $x \in A *'' M$, $(d \otimes 1 \otimes 1)(x) = -(d \otimes 1 \otimes 1)T(x) = -T(1 \otimes d \otimes 1)(x) = 0$. $\square$

Remark 7.5. Using Proposition 7.3 it is easy to compute the functors (7.4) for finitely generated abelian groups A. For this we need only to consider cyclic groups $\mathbb{Z}/n = A$ with the presentation $d_A = n: \mathbb{Z} = G \to \mathbb{Z} = F$. In this case we have $\mathbb{Z} \otimes_{\mathbb{Z}} M = M_e$ and $\mathrm{Hom}(\mathbb{Z}, M) = M_e$; therefore, the chain complexes (7.4) (1)–(4) can be expressed in terms of H, P in the quadratic $\mathbb{Z}$-module M. In particular, (7.4) (1), resp. (2), is given for $d_A = n$ by

$$\text{(1)} \qquad M_{ee} \xrightarrow{(P,\,-n)} M_e \oplus M_{ee} \xrightarrow{(n_*,\,nP)} M_e,$$

$$\text{(2)} \qquad M_{ee} \xleftarrow{(H,\,-n)} M_e \oplus M_{ee} \xleftarrow{(n_*,\,nH)} M_e,$$

where n_* is defined in Definition 2.1. In addition we can use the following formulae for the computation.

Proposition 7.6. *Let A be a finite abelian group and let $A^E = \mathrm{Ext}(A, \mathbb{Z})$. Then one has the natural isomorphisms*

$$\begin{aligned} &\mathrm{Ext}(A, M) = A^E \otimes M, && \mathrm{Hom}(A, M) = A^E * M, \\ &\mathrm{Hom}'(A, M) = A^E *' M, && \mathrm{Ext}'(A, M) = A^E \otimes' M, \\ &\mathrm{Hom}''(A, M) = A^E *'' M, && \mathrm{Ext}''(A, M) = A^E \otimes'' M. \end{aligned}$$

There is a non-natural isomorphism $A^E \cong A$.

Proof. Since A is finite we obtain a presentation of $\mathrm{Ext}(A, \mathbb{Z})$ by $d_A^*: F^\# = \mathrm{Hom}(F, \mathbb{Z}) \to G^\# = \mathrm{Hom}(G, \mathbb{Z})$. Using (5.1) we can replace $\mathrm{Hom}_{\mathbb{Z}}(F, M)$ by $F^\# \otimes_{\mathbb{Z}} M$. This way the chain complex (7.4)(3) for d_A is the same as the chain complex (7.4)(1) for d_A^*. This proves the left-hand side equations. □

Remark 7.7. The twelve functors in (7.4) evaluated on $A = \mathbb{Z}$ are given by:

$$\begin{aligned} &\mathbb{Z} \otimes M = M_e, && \mathbb{Z} *' M = 0, && \mathbb{Z} *'' M = 0, \\ &\mathbb{Z} * M = 0, && \mathbb{Z} \otimes' M = \ker H, && \mathbb{Z} \otimes'' M = \operatorname{cok} H, \\ &\mathrm{Ext}(\mathbb{Z}, M) = 0, && \mathrm{Hom}'(\mathbb{Z}, M) = \operatorname{cok} P, && \mathrm{Hom}''(\mathbb{Z}, N) = \ker P, \\ &\mathrm{Hom}(\mathbb{Z}, M) = M_e, && \mathrm{Ext}'(\mathbb{Z}, M) = 0, && \mathrm{Ext}''(\mathbb{Z}, M) = 0. \end{aligned}$$

Here H, P are the maps of the quadratic $\mathbb{Z}$-module M.

Theorem 7.8. *A short exact sequence*

$$0 \to K \xrightarrow{i} M \xrightarrow{q} N \to 0$$

of quadratic $\mathbb{Z}$-modules in $\mathcal{QM}(\mathbb{Z})$ induces the following four types of natural 9-term exact sequences.

(1) $$\begin{aligned} 0 &\to A *'' K \xrightarrow{i_*} A *'' M \xrightarrow{q_*} A *'' N \\ &\to A *' K \to A *' M \to A *' N \\ &\to A \otimes K \to A \otimes M \to A \otimes N \to 0, \end{aligned}$$

(2) $$\begin{aligned} 0 &\to A * K \to A * M \to A * N \\ &\to A \otimes' K \to A \otimes' M \to A \otimes' N \\ &\to A \otimes'' K \to A \otimes'' M \to A \otimes'' N \to 0, \end{aligned}$$

$$(3)\qquad \begin{aligned} &0 \to \mathrm{Hom}''(A, K) \to \mathrm{Hom}''(A, M) \to \mathrm{Hom}''(A, N) \\ &\to \mathrm{Hom}'(A, K) \to \mathrm{Hom}'(A, M) \to \mathrm{Hom}'(A, N) \\ &\to \mathrm{Ext}(A, K) \to \mathrm{Ext}(A, M) \to \mathrm{Ext}(A, N) \to 0, \end{aligned}$$

$$(4)\qquad \begin{aligned} &0 \to \mathrm{Hom}(A, K) \to \mathrm{Hom}(A, M) \to \mathrm{Hom}(A, N) \\ &\to \mathrm{Ext}'(A, K) \to \mathrm{Ext}'(A, M) \to \mathrm{Ext}'(A, N) \\ &\to \mathrm{Ext}''(A, K) \to \mathrm{Ext}''(A, M) \to \mathrm{Ext}''(A, N) \to 0. \end{aligned}$$

If the quadratic $\mathbb{Z}$-modules K, M, N are actually abelian groups, that is, if the short exact sequence in Theorem 7.8 lies in the subcategory $\mathcal{Ab}$ of $\mathcal{QM}(\mathbb{Z})$, see Definition 2.1, then the terms with an index $''$ above vanish so that in this case the 9-term exact sequences above coincide with the corresponding classical 6-term exact sequences of homological algebra.

Example 7.9. One has the short exact sequences

$$(1)\qquad 0 \to \mathbb{Z}^S \xrightarrow{i} \mathbb{Z}^P \to \mathbb{Z} \to 0,$$

$$(2)\qquad 0 \to \mathbb{Z}^S \xrightarrow{j} \mathbb{Z}^\Gamma \to \mathbb{Z}/2 \to 0,$$

with $i_e = (1, 1)$, $i_{ee} = 1$ and $j_e = 2$ and $j_{ee} = 1$. Hence we obtain by Theorem 7.8(1) via (1) the isomorphism $A *' \mathbb{Z}^S = A *' \mathbb{Z}^P$ and the short exact sequences

$$0 \to A \otimes \mathbb{Z}^S \to A \otimes \mathbb{Z}^P \to A \otimes \mathbb{Z} \to 0$$

which coincides with the top row of Remark 2.10(4). Moreover, by (2) we get the exact sequence ($A *'' \mathbb{Z}/2 = 0$)

$$\begin{aligned} &0 \to A *' \mathbb{Z}^S \to A *' \mathbb{Z}^\Gamma \xrightarrow{\sigma} A * \mathbb{Z}/2 \\ &\xrightarrow{0} A \otimes \mathbb{Z}^S \to A \otimes \mathbb{Z}^\Gamma \xrightarrow{\sigma} A \otimes \mathbb{Z}/2 \to 0, \end{aligned}$$

which is a union of two short exact sequences. The second part coincides with the bottom row of Remark 2.10(4) and $A *' \mathbb{Z}^\Gamma = R(A)$ is given by Eilenberg–Mac Lane's functor R. Compare the exact sequence in Example 10.4. There are indeed many further interesting applications of the 9-terms exact sequences above.

Proof of Theorem 7.8. We first prove (1). For this we observe that the short exact sequence of quadratic $\mathbb{Z}$-modules in Theorem 7.8 induces a short exact sequence of chain complexes

$$(*) \qquad 0 \to (\{-\} \otimes_{\mathbb{Z}} K)_* d_A \to (\{-\} \otimes_{\mathbb{Z}} M)_* d_A \to (\{-\} \otimes_{\mathbb{Z}} N)_* d_A \to 0.$$

Indeed this is short exact since F and G in (7.4)(1) are free abelian. To see this we use Definition 2.4(3), Proposition 7.3 and Remark 7.7. Now the long exact Bockstein sequence of homology groups applied to $(*)$ yields (1). In a similar way we obtain the other 9-term exact sequences. $\square$

Remark 7.10. It is also of interest to consider the natural quadratic cross effect sequences derived from the 9-term exact sequences above. For example, Theorem 7.8(1) and Proposition 7.3 yield the natural exact sequence

$$\begin{aligned} &0 \to A * B * K_{ee} \to A * B * M_{ee} \to A * B * N_{ee} \\ &\to \mathrm{Trp}(A, B, K_{ee}) \to \mathrm{Trp}(A, B, M_{ee}) \to \mathrm{Trp}(A, B, N_{ee}) \\ &\to A \otimes B \otimes K_{ee} \to A \otimes \mathrm{B} \otimes M_{ee} \to A \otimes \mathrm{B} \otimes N_{ee} \to 0. \end{aligned}$$

A short exact sequence of abelian groups induces as well certain exact sequences for quadratic tensor products, this is discussed in Appendix B, see Theorem B.9.

8. Quadratic homotopy functors

We introduce additive categories of homotopy abelian co-H-groups and H-groups respectively and we describe quadratic functors on these categories. The functors are given by homotopy groups, homology groups, and cohomology groups respectively.

Let $\mathcal{CW}$-*spaces*$^*/\simeq$ be the homotopy category of CW-spaces with basepoint $*$; the set of morphisms $X \to Y$ in this category is the set of homotopy classes $[X, Y]$. We write $\dim(Y) \leq m$ if there is a homotopy equivalence $Y \simeq X$ where X is an m-dimensional CW-complex. Moreover, we write $\mathrm{hodim}(Y) \leq m$ if $\pi_i(Y) = 0$ for $i > m$. Let $\mathcal{A}_n^k$, resp. $\mathcal{B}_n^k$ be the full subcategories of $\mathcal{CW}$-*spaces*$^*/\simeq$ consisting of $(n-1)$-connected spaces X with $\dim(X) \leq n + k$, resp. $\mathrm{hodim}(X) \leq n + k$. Let G be an abelian group. An *Eilenberg–Mac Lane space* $K(G, n)$ is a CW-space with $\pi_n(K(G, n)) = G$ and $\pi_j K(G, n) = 0$ for $j \neq n$. A *Moore space* $M(G, n)$ is a simply connected CW-space with homology groups $H_n M(G, n) = G$ and $H_j M(G, n) = 0$, $n \neq j \geq 1$. We clearly have $\mathrm{hodim}\, K(G, n) \leq n$ and $\dim M(G, n) \leq n + 1$.

Definition 8.1. Let $\mathcal{HA}$ and $co\mathcal{HA}$ be the following subcategories of $\mathcal{CW}$-*spaces*$^*/\simeq$. Objects in $\mathcal{HA}$ are homotopy abelian H-groups and morphism are H-maps. The objects in $co\mathcal{HA}$ are homotopy abelian co-H-groups and morphisms are co-H-maps. Let $\mathscr{H}\mathscr{A}_n$, resp. $co\mathscr{H}\mathscr{A}_n$ be the full subcategories consisting of $(n-1)$-connected objects.

For example, a double loop space $\Omega^2 Y$ and a double suspension $\Sigma^2 Y$ are objects in $\mathscr{H}\mathscr{A}$ and $co\mathscr{H}\mathscr{A}$ respectively. This shows that one has full inclusions

$$\mathscr{A}_n^k \subset co\mathscr{H}\mathscr{A}_n \quad \text{and} \quad \mathscr{B}_n^k \subset \mathscr{H}\mathscr{A}_n \qquad \text{for } k < n-1. \tag{8.1}$$

All categories in (8.1) are additive categories; the biproduct in $co\mathscr{H}\mathscr{A}$ is given by the one-point union $X \vee Y$ of spaces and the biproduct in $\mathscr{H}\mathscr{A}$ is given by the product $X \times Y$ of spaces. For a CW-space K let π_m^K and π_K^m be the homotopy functors defined by

$$\pi_m^K(X) = [\Sigma^m K, X] \quad \text{and} \quad \pi_K^m(X) = [X, \Omega^m K]. \tag{8.2}$$

As usual we have $\pi_m^K(X) = \pi_m(X)$ if $K = S^0$ is the 0-sphere and we have $\pi_K^m(X) = H^k(X, G)$ if $K = K(G, m+k)$. The sets in (8.2) are groups, resp. abelian groups, for $m = 1$, resp. $m \geq 2$. Using the homotopy functors (8.2) and the homology and cohomology functors we obtain the following four functors:

$$\pi_m^K : co\mathscr{H}\mathscr{A}_n \to \mathscr{Ab} \quad \text{with } \dim(\Sigma^m K) < 3n-2, \tag{8.3a}$$

$$\pi_K^m : \mathscr{H}\mathscr{A}_n^{\mathrm{op}} \to \mathscr{Ab} \quad \text{with } \operatorname{hodim}(\Omega^m K) < 3n, \tag{8.3b}$$

$$H^m(-, G) : \mathscr{H}\mathscr{A}_n^{\mathrm{op}} \to \mathscr{Ab} \quad \text{with } m < 3n, \tag{8.3c}$$

$$H_m(-, G) : \mathscr{H}\mathscr{A}_n \to \mathscr{Ab} \quad \text{with } m < 3n. \tag{8.3d}$$

The functor (8.3c) is a special case of (8.3b) when we set $K = K(G, m+k)$. The conditions on the right-hand side describe the *meta stable range* of these functors. It is well known that in this range the functors are quadratic. In the *stable range* (given by $\dim(\Sigma^m K) < 2n-1$, $\operatorname{hodim}(\Omega^m K) < 2n$, resp. $m < 2n$) the functors are additive.

We now consider the cross effects and the structure maps H, P, T in Definition 2.4 for the quadratic functors in (8.3). For suspensions $X = \Sigma X'$, $Y = \Sigma Y'$, the Hilton–Milnor theorem shows

$$\pi_m^K(\Sigma X' \wedge Y') \cong \pi_m^K(X \mid Y). \tag{8.4}$$

Here the isomorphism is induced by the injection $\pi_m^K([i_1, i_2])$ where $[i_1, i_2] : \Sigma X' \wedge Y' \to X \vee Y$ is the Whitehead product map. Using (8.4) as an identification the map T coincides with $-(\Sigma T_{21})_*$ where $T_{21} : X' \wedge Y' \to Y' \wedge X'$ is the interchange map. Moreover, the maps

$$\pi_m^K\{\Sigma X'\} = \left(\pi_m^K(\Sigma X') \xrightarrow{H} \pi_m^K(\Sigma X' \wedge X') \xrightarrow{P} \pi_m^K(\Sigma X')\right),$$

given by (8.3), coincide with the James–Hopf invariant $H = \gamma_2$ and the Whitehead product map $P = [1, 1]_*$ where $1 = 1_X$ is the identity. These maps H and P are exactly the operators which appear in the classical EHP-sequence of homotopy theory. Next we obtain the cross effects of the functors (8.3b)–(8.3d) by canonical isomorphisms

$$\begin{aligned}&\pi_K^m(X \wedge Y) \cong \pi_K^m(X \mid Y),\\ &H^m(X \wedge Y, G) \cong H^m(X \mid Y, G),\\ &H_m(X \wedge Y, G) \cong H_m(X \mid Y, G)\end{aligned} \tag{8.5}$$

which are readily obtained by the cofiber sequence $X \vee Y \to X \times Y \to X \wedge Y$. For (8.3b) and (8.3c) the maps H, P, T correspond to $H = (H\mu)^*$, $P = \Delta^*$, $T = (T_{21})^*$, where $\Delta: X \to X \wedge X$ is the reduced diagonal and where $H\mu: \Sigma X \wedge X \to \Sigma X$ is the Hopf-construction of the H-space multiplication $\mu = r_1 + r_2: X \times X \to X$. In (8.3d) we get $H = \Delta_*$, $P = (H\mu)_*$ and $T = (T_{21})_*$. For the definition of $H\mu$ see for example [6, Chapter II, 15.15]. For $(H\mu)^*$ and $(H\mu)_*$ we use the canonical suspension isomorphisms $\pi_K^{m-1}(\Sigma X) = \pi_K^m(X)$ and $H_{m+1}(\Sigma X, G) = H_m(X, G)$.

9. Homotopy groups of Moore spaces

We describe a six-term exact sequence for the homotopy groups of Moore spaces which is useful for computation in the metastable range of these groups. As an application we obtain a new homotopy invariant $\tau(X)$ of an $(n-1)$-connected $(2n+1)$-dimensional closed manifold X.

Let $\mathscr{R} \subset co\mathscr{H}\mathscr{A}_n$ be a small subringoid consisting of suspensions $X = \Sigma X'$. A CW-space U gives us the $\mathscr{R}^{\mathrm{op}}$-module (= additive functor)

$$[\mathscr{R}, U]: \mathscr{R}^{\mathrm{op}} \to \mathscr{A}b$$

which carries $X \in \mathscr{R}$ to the abelian group $[X, U]$. The quadratic $\mathscr{R}$-module $\pi_m^K\{\mathscr{R}\}$ associated to (8.3a) and the tensor product of Definition 3.1 can be used for the natural homomorphism ($\dim \Sigma^m K < 3n - 2$)

$$\lambda: [\mathscr{R}, U] \otimes_{\mathscr{R}} \pi_m^K\{\mathscr{R}\} \to \pi_m^K(U) \tag{9.1}$$

which we call a *tensor approximation* of $\pi_m^K(U)$. For $a \in [\Sigma X', U]$, $b \in [\Sigma Y', U]$, $(\Sigma X', \Sigma Y' \in \mathscr{R})$, and for $\alpha \in [\Sigma^m Y, \Sigma X']$, $\beta \in [\Sigma^m Y, \Sigma X' \wedge Z']$ we define λ by $\lambda(a \otimes \alpha) = a \circ \alpha$ and $\lambda([a, b] \otimes \beta) = [a, b] \circ \beta$ where $[a, b]$ is the Whitehead product. The image of λ is the subgroup generated by all compositions

$$\Sigma^m Y \xrightarrow{\alpha} X_1 \vee \cdots \vee X_k \xrightarrow{a} U$$

with $X_i \in \mathscr{R}$, $k \geq 1$. The map α is in the metastable range. The composition $a \circ \alpha$, however, needs not to be in the metastable range.

Lemma 9.1. *λ in (9.1) is a well-defined natural homomorphism. Moreover, λ is an isomorphism if $U = X_1 \vee \cdots \vee X_k$ with $X_i \in \mathscr{R}$ and if $[X, X_i] \subset \mathscr{R}(X, X_i)$ for all $i = 1, \ldots, k$ and $X \in \mathscr{R}$.*

Proof. The lemma is a consequence of the distributivity laws [5] and of Proposition 4.3. □

Remark 9.2. A natural description of the homotopy group $\pi_m^K M(A, n)$ of the Moore space $M(A, n)$ can be obtained by the tensor approximation (9.1). For this we need to consider *elementary Moore spaces* $M(\mathbb{Z}, n) = S^n$ or $M(\mathbb{Z}/p^i, n)$, p = prime. Let $\mathscr{R}$ be the full homotopy category consisting of elementary Moore spaces. Then (9.1) yields the natural homomorphism, $n \geq 3$,

$$\lambda : [\mathscr{R}, M(A, n)] \otimes_{\mathscr{R}} \pi_m^K\{R\} \to \pi_m^K M(A, n)$$

which is an isomorphism if A is finitely generated. This follows from Lemma 9.1.

We now consider an example of λ in (9.1) where $\mathscr{R} \cong \mathbb{Z}$ is the full subcategory consisting only of the sphere S^n and where $U = M(A, n)$. Then $\pi_m^K\{\mathscr{R}\}$ is just the quadratic $\mathbb{Z}$-module

$$\pi_m^K\{S^n\} = \left(\pi_m^K(S^n) \xrightarrow{H} \pi_m^K(S^{2n-1}) \xrightarrow{P} \pi_m^K(S^n) \right)$$

which is defined by $F = \pi_m^K(-)$ as in (8.3c); here H is the Hopf invariant and $P = [1, 1]_*$ as in (8.4). Now (9.1) gives us the natural homomorphism

$$\lambda : A \otimes_{\mathbb{Z}} \pi_m^K\{S^n\} \to \pi_m^K M(A, n) \tag{9.2}$$

which is an isomorphism if A is a free abelian group (here A need not to be finitely generated). It is an old result of Hopf that $\pi_3\{S^2\} \cong \mathbb{Z}^\Gamma = (\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z})$. Therefore we derive from Remark 9.2 the natural homomorphism $\lambda : \Gamma(A) = A \otimes \mathbb{Z}^\Gamma \cong \pi_3 M(A, 2)$ which is actually an isomorphism for all abelian groups A, see [37], Definition 2.9 and Proposition 4.5. In general the map λ in (9.2) is not an isomorphism. Let $S \subset \pi_m^K M(A, n)$ be the subgroup generated by all compositions $\Sigma^m K \to S^n \vee \cdots \vee S^n \to M(A, n)$ and let

$${}_\lambda\pi_m^K M(A, n) = \pi_m^K M(A, n)/S$$

be the quotient group. For $\dim \Sigma^m K < 3n - 2$ this is the cokernel of λ in (9.2). Now λ is embedded in the following exact sequence which shows the relevance of the corresponding derived functors in (7.4).

Theorem 9.3. *For* $\dim(\Sigma^m K) < 3n - 2$ *there is a natural exact sequence*

$$0 \to A *' \pi_m^K\{S^n\} \to {}_\lambda\pi_{m+1}^K M(A,n) \to A *'' \pi_{m-1}^K\{S^n\}$$
$$\xrightarrow{\partial} A \otimes \pi_m^K\{S^n\} \xrightarrow{\lambda} \pi_m^K M(A,n) \xrightarrow{q} {}_\lambda\pi_m^K M(A,n) \to 0,$$

where q is the quotient map.

Proof. Theorem 9.3 is a special case of (2.7) in [8]. For this let $X_1 \xrightarrow{d} X_0 \twoheadrightarrow A$ be a short free resolution of the abelian group A and let $g: M(X_1,n) \to M(X_0,n)$ be a map which induces d. The mapping cone of g is the Moore space $M(A,n) = C_g$. Using the isomorphism λ in (9.2) (where we replace A by X_1 and X_0 respectively) we obtain isomorphisms

$$H_0\{\pi_m^K\}_*(g) = A \otimes \pi_m^K\{S^n\},$$
$$H_1\{\pi_m^K\}(g) = A *' \pi_m^K\{S^n\},$$
$$H_2\{\pi_m^K\}_*(g) = A *'' \pi_m^K\{S^n\}.$$

Compare the definition in (7.4). Now it is easy to see that i in [8, (2.7)] corresponds to λ in the theorem. Therefore the theorem is just a special case of [8, (2.7)]. □

Corollary 9.4. *For $m \le \min(2n, 3n-3)$ one has the natural short exact sequence*

$$0 \to A \otimes \pi_m\{S^n\} \xrightarrow{\lambda} \pi_m M(A,n) \to A *' \pi_{m-1}\{S^n\} \to 0$$

and the isomorphism ${}_\lambda\pi_{m+1} M(A,n) \cong A *' \pi_m\{S^n\}$.

Proof. Since $\pi_{2n-1}S^{2n-1} = \mathbb{Z}$ we see that $A *'' \pi_{m-1}\{S^n\} = 0$ for $m \le 2n$, compare Proposition 7.4. Whence Corollary 9.4 is a consequence of Theorem 9.3. In the stable range $m < 2n - 1$ the sequence of Corollary 9.4 is well known (see for example [2]); in this case we have $A \otimes \pi_m\{S^n\} = A \otimes \pi_m S^n$ and $A *' \pi_{m-1}\{S^n\} = A * \pi_{m-1} S^n$, see Remark 7.1. □

Next consider the cross effects of the exact sequence in Theorem 9.3. For this let $M(A \mid B, n) = M(A,n) \wedge M(B, n-1)$ and let ${}_\lambda\pi_m^K(A \mid B, n) = \pi_m^K M(A \mid B, n)/S'$ where S' is the subgroup generated by all compositions $\Sigma^m K \to S^{2n-1} \to M(A \mid B, n)$.

Corollary 9.5. *For* $\dim(\Sigma^m K) < 3n - 2$ *there is a natural exact sequence*

$$0 \to \operatorname{Trp}(A, B, \pi_m^K S^{2n-1}) \to {}_\lambda\pi_{m+1}^K M(A \mid B, n) \to A * B * \pi_{m-1}^K S^{2n-1}$$

$$\xrightarrow{\partial} A \otimes B \otimes \pi_m^K S^{2n-1} \to \pi_m^K M(A \mid B, n) \to {}_\lambda\pi_m^K M(A \mid B, n) \to 0.$$

Here Trp is the triple torsion product of Mac Lane [21], see also Proposition 7.3(3). Corollary 9.5 is the "cross effect sequence" of Theorem 9.3 obtained by the formulas of Proposition 7.3(3). It is an interesting problem to compute the boundary operators ∂ in Theorem 9.3 and Corollary 9.5 only in terms of some structure of the homotopy groups $\pi_i^K(S^j)$ of spheres, in partiuclar if $K = S^0$.

Remark 9.6. There are many papers in the literature concerning the homotopy groups of Moore spaces $\pi_m M(A, n)$, see for example [13, 27, 33]. We here are mainly interested in the functorial properties of $\pi_m M(A, n)$, $m < 3n - 2$, which are not so well understood; an early approach in this direction is due to Barratt [2] for $m < 2n - 1$.

The functorial properties of the groups $\pi_m M(A, n)$ are of special interest for the homotopy classification of manifolds and Poincaré-complexes respectively. Let P_n^k be the class of $(n - 1)$-connected $(2n + k)$-dimensional Poincaré-complexes.

Examples 9.7. Let $n \geq 2$. For $X \in P_n^0$ there is a homotopy invariant

$$\varepsilon(X) \in H_n(X) \otimes \pi_{2n-1}\{S^n\}$$

where $H_n X$ is a finitely generated free abelian group. In fact, X is the mapping cone $X \simeq C_f$ of a map $f: S^{2n-1} \to M(H_n X, n)$ and $\varepsilon(X) = \lambda^{-1}(f)$ is given by the isomorphism λ in (9.2). Whence $\varepsilon(X)$ is a complete homotopy invariant of X, that is for $X, Y \in P_n^0$ there is an orientation preserving homotopy equivalence $X \simeq Y$ if there is an isomorphism $\varphi: H_n X \cong H_n Y$ with $(\varphi \otimes 1)\varepsilon(X) = \varepsilon(Y)$. We can write the invariant $\varepsilon(X)$ in terms of the cohomology $H^n(X)$ as follows. Since $H_n(X) = \operatorname{Hom}(H^n(X); \mathbb{Z})$ we have by (5.1) the isomorphism

$$\chi: H_n(X) \otimes \pi_{2n-1}\{S^n\} \cong \operatorname{Hom}(H^n(X), \pi_{2n-1}\{S^n\})$$

Therefore, $\chi\varepsilon(X) = (\alpha_e, \alpha_{ee})$ is a quadratic form with $\alpha_e: H^n(X) \to \pi_{2n-1} S^n$ and with $\alpha_{ee}: H^n(X) \times H^n(X) \to \pi_{2n-1} S^{2n-1} \cong \mathbb{Z}$. Here α_{ee} is just the cup product pairing in X. Moreover, $\alpha_e = \Psi$ is exactly the cohomology operation considered by Kervaire and Milnor in [20, 8.2] (the formula there is equivalent to the fact that (α_e, α_{ee}) is a quadratic form, compare the first equation in Definition 5.1(2)).

Example 9.8. For $X \in P_n^1$ $(n \geq 2)$ we define a new homotopy invariant

$$\tau(X) \in H_n(X) *' \pi_{2n-1}\{S^n\}$$

which we call the *torsion-invariant* of X. We obtain $\tau(X)$ by a homotopy equivalence $X \simeq C_f$ where $f: S^{2n} \to M(H_{n+1}X, n+1) \vee M(H_nX, n)$. Let $r_2 f \in \pi_{2n} M(H_nX, n)$ be given by the retraction r_2 and let $\tau(X)$ be the image of $r_2 f$ under the homomorphism

$$\pi_{2n} M(H_nX, n) \twoheadrightarrow {}_\lambda\pi_{2n} M(H_nX, n) \cong H_n(X) *' \pi_{2n-1}\{S^n\}$$

given by Corollary 9.4. One can check that an orientation preserving map $v: X \to Y$ with $X, Y \in P_n^1$ satisfies

$$(H_n(v) *' 1)(\tau(X)) = \tau(Y)$$

so that $\tau(X)$ is a well-defined homotopy invariant. For $n \geq 3$ the exact sequence of Corollary 9.4 can be used for the computation of all possible f which yield the same torsion invariant. This yields a kind of homotopy classification of objects in P_n^1, (using different invariants such a classification is intensively studied in [17, 28, 30, 31, 36]).

Examples 9.9. (*Examples of computations*) Table 2 shows some examples of the quadratic $\mathbb{Z}$-modules $\pi_m\{S^n\}$ where we use the notation for indecomposable quadratic $\mathbb{Z}$-modules of Table 1 and Definition 2.9. These examples can be deduced from Toda's computations [34]. In the list we denote a cyclic group $\mathbb{Z}/n$ simply by n and we denote a direct sum $\mathbb{Z}/n \oplus \mathbb{Z}/m$ by $n \oplus m$. Moreover, (n, m) and (n, m, r) are the greatest common divisors.

The quadratic $\mathbb{Z}$-module $\mathbb{Z}_2^P$ (see $(n, m) = (4, 7)$) is given by

$$\mathbb{Z}_2^P = \left(\mathbb{Z} \otimes \mathbb{Z}/4 \xrightarrow{(1,0)} \mathbb{Z} \xrightarrow{(2,-1)} \mathbb{Z} \oplus \mathbb{Z}/4\right)$$

and ε_k in this line is

$$\varepsilon_k = \begin{cases} 2 & k \equiv 0(4),\ k \not\equiv 0(8), \\ 4 & k \equiv 0(8), \\ 0 & \text{otherwise.} \end{cases}$$

Table 2

n, m	$\pi_m\{S^n\}$	$\mathbb{Z}/k \otimes \pi_m\{S^n\}$	$\mathbb{Z}/k *' \pi_m\{S^n\}$	$\mathbb{Z}/k *'' \pi_m\{S^n\}$
2, 3	$\mathbb{Z}^r$	$(k^2, 2k)$	$(k, 2)$	0
3, 5	$\mathbb{Z}^A \oplus 2$	$(k, 2)$	$(k, 2) \oplus k$	0
3, 6	$H(2, 1) \oplus 3$	$(k, 12)$	$(k, 12) \oplus (k, 2)$	$(k, 2)$
4, 7	$\mathbb{Z}_2^P \oplus 3$	$(k^2, 2k) \oplus \varepsilon_k \oplus (k, 3)$	$(k, 24)$	0
4, 8		0	0	0 $k \equiv 1(2)$
	$(\mathbb{Z}/2)^P$	2	2	0 $k \equiv 2(4)$
4, 9		$2 \oplus 2$	$2 \oplus 2$	0 $k \equiv 0(4)$
5, 9	$P(1)$	$(k, 2)$	$(k^2, 2k)$	0
5, 10	$(\mathbb{Z}/2)^S$	$(k, 2)$	$(k, 2)$	0
5, 11	$(\mathbb{Z}/2)^A \oplus 2$	$(k, 2)$	$(k, 2) \oplus (k, 2)$	$(k, 2)$
5, 12	$H(1, 3) \oplus 15 \oplus (\mathbb{Z}/3)^A$	$(k, 2) \oplus (k, 15)$	$(k, 2) \oplus (k, 3) \oplus (k, 8) \oplus (k, 15)$	$(k, 3) \oplus (k, 8)$
6, 11	$\mathbb{Z}^S$	k	0	0
6, 12	$(\mathbb{Z}/2)^A \oplus 2$	$(k, 2)$	$(k, 2) \oplus (k, 2)$	$(k, 2)$
6, 13	$H(2, 1) \oplus 15$	$(k, 60)$	$(k, 60) \oplus (k, 2)$	$(k, 2)$
6, 14	$(\mathbb{Z}/8)^r \oplus 2 \oplus (\mathbb{Z}/3)^S$	$(k, 2) \oplus (k^2, 2k, 24)$	$(k, 2) \oplus (k, 2) \oplus (k^2, 2k, 24)$	$(k, 2)$
6, 15	$2 \oplus 2 \oplus 2$	$(k, 2) \oplus (k, 2) \oplus (k, 2)$	$(k, 2) \oplus (k, 2) \oplus (k, 2)$	0

Moreover, for $(n, m) = (4, 8), (4, 9)$ we use $(\mathbb{Z}/2)^P = [\mathbb{Z}^P] \otimes \mathbb{Z}/2$ as defined in Definition 2.1.

The computation of the groups in this table is readily obtained by Remark 7.5. Combining the groups in the list with the exact sequences of Theorem 9.3 and Corollary 9.4 we immediately get the following short exact sequences:

(1) $$\mathbb{Z}/(k, 12) \rightarrowtail \pi_6 M(\mathbb{Z}/k, 3) \twoheadrightarrow \mathbb{Z}(k, 2) \oplus \mathbb{Z}/k,$$

(2) $$\left.\begin{array}{ll} k \equiv 1(2) & 0 \\ k \equiv 2(4) & \mathbb{Z}/2 \\ k \equiv 0(4) & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \end{array}\right\} \rightarrowtail \pi_8 M(\mathbb{Z}/k, 4) \twoheadrightarrow \mathbb{Z}/(k, 24),$$

(3) $$\left.\begin{array}{ll} k \equiv 1(2) & 0 \\ k \equiv 2(4) & \mathbb{Z}/2 \\ k \equiv 0(4) & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \end{array}\right\} \rightarrowtail \pi_9 M(\mathbb{Z}/k, 4) \twoheadrightarrow \left\{\begin{array}{l} 0 \\ \mathbb{Z}/2, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, \end{array}\right.$$

(4) $$\mathbb{Z}/(k, 2) \rightarrowtail \pi_{10} M(\mathbb{Z}/k, 5) \twoheadrightarrow \mathbb{Z}/(2k, k^2),$$

(5) $$\mathbb{Z}/(k, 2) \rightarrowtail \pi_{11} M(\mathbb{Z}/k; 5) \twoheadrightarrow \mathbb{Z}/(k, 2).$$

By a result of Sasao [27] the sequence (1) is non-split only for $k \equiv 0(2)$ and $k/(k, 12) \equiv 1(2)$; in this case one has $\pi_6 M(\mathbb{Z}/k, 3) = \mathbb{Z}/2 \oplus \mathbb{Z}/2k \oplus \mathbb{Z}/(k, 12)/2$. Moreover, Tipple [33] showed that (3) is split and that (4) is non-split only for $k \equiv 2(4)$. Finally we deduce $\pi_{12} M(\mathbb{Z}/k, 6) = \mathbb{Z}/(k, 12)$ from Table 2. We leave it to the reader to describe further examples for the exact sequences.

10. Homology of Eilenberg–Mac Lane spaces

We describe a six-term sequence for the metastable homology groups of Eilenberg–Mac Lane complexes. This sequence is a kind of Eckmann–Hilton dual of the corresponding exact sequence for metastable homotopy groups of Moore spaces in Section 9. Moreover, using the operators in Whitehead's certain exact sequence we obtain a map which carries the homotopy groups of Moore spaces to the homology groups of Eilenberg–Mac Lane spaces.

Let $\mathscr{R} \subset \mathscr{H}\mathscr{A}_n$ be a small subringoid, see Definition 8.1. A homotopy abelian H-space U, $U \in \mathscr{H}\mathscr{A}$, gives us the $\mathscr{R}^{\mathrm{op}}$-module

$$[\mathscr{R}, U]' : \mathscr{R}^{\mathrm{op}} \to \mathscr{A}b$$

which carries $X \in \mathscr{R}$ to the abelian group of H-maps $[X, U]' = \mathscr{H}\mathscr{A}(X, U)$ which is a subgroup of $[X, U]$. The quadratic $\mathscr{R}$-module $H_m\{\mathscr{R}, G\}$ associated to (8.3d) and the tensor product of Definition 3.1 yield the natural homomorphism $(m < 3/n)$

$$\lambda : [\mathscr{R}, U]' \otimes_{\mathscr{R}} H_m\{\mathscr{R}, G\} \to H_m(U, G) \tag{10.1}$$

as follows. For $a \in [X, U]'$, $b \in [Y, U]'$, $\alpha \in H_m(X, G)$, $\beta \in H_m(X \wedge Y, G)$ let $\lambda(a \otimes \alpha) = a_*(\alpha)$ and $\lambda([a, b] \otimes \beta) = H(\mu)_*(a \wedge b)_*(\beta)$, compare (8.5). The image of λ is the subgroup of $H_m(U, G)$ generated by all elements $\alpha_*(a)$ where $\alpha : X_1 \times \cdots \times X_k \to U$ is an H-map, $X_i \in \mathscr{R}$, $k \geq 1$, and where $a \in H_m(X_1 \times \cdots \times X_k, G)$.

Lemma 10.1. λ in (10.1) *is a well-defined natural homomorphism. Moreover* λ *is an isomorphism if* $U = X_1 \times \cdots \times X_k$, $X_i \in \mathscr{R}$ *for* $i = 1, \ldots, k$ *and if* $\mathscr{R}$ *is a full subringoid of* $\mathscr{H}\mathscr{A}_n$.

Proof. Similarly as in Lemma 9.1, the lemma is a consequence of Proposition 4.3. □

Remark 10.2. A natural description of $H_m(K(A, n), G)$, $m < 3n$, can be obtained by (10.1). For this let $\mathscr{R}$ be the full homotopy category consisting of elementary Eilenberg–Mac Lane spaces $K(\mathbb{Z}, n)$ or $K(\mathbb{Z}/p^i, n)$, $p =$ prime. Then (10.1) yields the natural homomorphism $(n \geq 2)$

$$\lambda : [\mathscr{R}, K(A, n)] \otimes_{\mathscr{R}} H_m\{\mathscr{R}, G\} \xrightarrow{\cong} H_m(K(A, n), G)$$

which is an isomorphism for all $A \in \mathscr{A}b$. This follows essentially from Lemma 10.1, compare Corollary 4.4. We clearly have $[\mathscr{R}, K(A, n)] = [\mathscr{R}, K(A, n)]'$.

We now consider a special case of λ in (10.1). For this let $\mathscr{R} \cong \mathbb{Z}$ be the full subcategory consisting only of $K(\mathbb{Z}, \mathrm{n})$ and let $U = K(A, n)$. Then $H_m\{\mathscr{R}, G\}$ is the quadratic $\mathbb{Z}$-module (see (8.5))

$$H_m^G\{n\} = \left(H_m(K(\mathbb{Z}, n), G) \xrightarrow{H} H_m(K(\mathbb{Z}, n), G) \xrightarrow{P} H_m(K(\mathbb{Z}, n), G) \right)$$

and we get by (10.1) the natural homomorphism

$$\lambda : A \otimes_{\mathbb{Z}} H_m^G\{n\} \to H_m(K(A, n), G) \tag{10.2}$$

which is an isomorphism if A is free abelian; here A need not to be finitely generated. In fact, λ is the tensor approximation of the functor $\mathscr{Ab} \to \mathscr{Ab}$ which carries A to $H_m(K(A, n), G)$, compare Proposition 4.5. For $G = \mathbb{Z}$ we set $H_m\{n\} = H_m^{\mathbb{Z}}\{n\}$. Since $K(\mathbb{Z}, 2) = \mathbb{C}P_\infty$ we readily see that $H_4\{2\} \cong \mathbb{Z}^\Gamma$. Therefore we derive from (10.2) the natural homomorphism $\lambda : \Gamma(A) = A \otimes \mathbb{Z}^\Gamma \cong H_4 K(A, 2)$ which is actually an isomorphism for all A, compare [16]. Table 3 shows some examples of quadratic $\mathbb{Z}$-modules $H_m\{n\}$. We use in this list the notation for indecomposable quadratic $\mathbb{Z}$-modules of Table 1 and Remark 2.10; the examples can be deduced from the computations in [16].

In general the map λ in (10.2) is not an isomorphism. As an analogue of Theorem 9.3 we obtain the following result. Again we use the derived functors in (7.4).

Theorem 10.3. *Let $m \leq 3n - 3$. Then there is a natural map $\kappa : H_m(K(A, n-1), G) \to A * H_m^G\{n\}$ such that ${}_\kappa H_m(K(A, n-1), G) = \ker(\kappa)$ is embedded in the natural exact sequence*

$$\begin{aligned} 0 \leftarrow A \otimes' H_m^G\{n\} &\leftarrow {}_\kappa H_{m-1}(K(A, n-1), G) \leftarrow A \otimes'' H_{m+1}^G\{n\} \\ &\leftarrow A * H_m^G\{n\} \xleftarrow{\kappa} H_m(K(A, n-1), G) \xleftarrow{i} {}_\kappa H_m(K(A, n-1), G) \leftarrow 0, \end{aligned}$$

where i is the inclusion.

In the stable range $m < 2n - 2$ this yields just the short exact sequence

$$A * H_m(K(\mathbb{Z}, n), G) \overset{\kappa}{\twoheadleftarrow} H_m(K(A, n-1), G) \leftarrowtail A \otimes H_{m+1}\,(K(\mathbb{Z}, n), G) \tag{10.3}$$

which is a kind of Eckmann–Hilton dual of the sequence in Corollary 9.4. Using the formulas in Proposition 7.3 it is easy to obtain the exact "cross effect sequence" of Theorem 9.3, this is a sequence of a similar nature as in Corollary 9.5.

Proof. The theorem is a special case of (3.12) in [8]. For this let $X_1 \xrightarrow{d} X_0 \twoheadrightarrow A$ be a short free resolution of A and let $g: K(X_1, n) \to K(X_0, n)$ be a map which induces d. Then the fiber of g is the Eilenberg–Mac Lane space $K(A, n-1) = P_g$. Therefore we can apply [8, (3.12)]. Using the isomorphism λ in (10.2) (where we replace A by X_0 and X_1 respectively) we get the isomorphisms

$$H^0\{H_m^G\}_*(g) \cong A * H_m^G\{n\},$$

$$H^1\{H_m^G\}_*(g) \cong A \otimes' H_m^G\{n\},$$

$$H^2\{H_m^G\}_*(g) \cong A \otimes'' H_m^G\{n\}.$$

Compare the definition in (7.4). Whence the theorem is just a special case of [8, (3.12)]. □

Example 10.4. We describe some applications of Theorem 10.3 where we use Table 3. Since $H_7\{4\} = 0$ we obtain the isomorphism

$$\begin{aligned} A \otimes' \mathbb{Z}^\Gamma \oplus A \otimes \mathbb{Z}/3 &= A \otimes' H_8(4) \cong {}_\kappa H_7 K(A, 3) \\ &= H_7 K(A, 3) \cong \Omega A \oplus A \otimes \mathbb{Z}/3 \end{aligned}$$

which corresponds to the isomorphism $A \otimes' \mathbb{Z}^\Gamma \cong \Omega A$. Since $H_7\{3\} = \mathbb{Z}/3$ we have $A \otimes'' H_7\{3\} = 0$ so that ${}_\kappa H_5 K(A, 2) \cong A \otimes' \mathbb{Z}^A$ where $\mathbb{Z}^A = H_6\{3\}$. Moreover we have $H_4\{3\} = 0$ so that ${}_\kappa H_4 K(A, 2) = H_4 K(A, 2) \cong \Gamma(A)$. Therefore, we derive from Theorem 10.3 the exact sequence

$$A \otimes \mathbb{Z}/2 \twoheadleftarrow \Gamma(A) \leftarrowtail A \otimes'' \mathbb{Z}^A \xleftarrow{0} A * \mathbb{Z}/2 \overset{\kappa}{\twoheadleftarrow} R(A) \leftarrowtail A \otimes' \mathbb{Z}^A$$

Table 3

m	n	$H_m\{n\}$	$H_m(K(A, n))$
3	2	0	0
4	2	$\mathbb{Z}^\Gamma$	$\Gamma(A)$
5	2	0	$R(A)$
5	3	$\mathbb{Z}/2$	$\mathbb{Z}/2 \otimes A$
6	3	$\mathbb{Z}^A$	$\mathbb{Z}/2 * A \oplus \Lambda^2(A)$
7	3	$\mathbb{Z}/3$	$\mathbb{Z}/3 \otimes A \oplus \Omega(A)$
8	3	$(\mathbb{Z}/2)^\otimes$	$\mathbb{Z}/3 * A \oplus (\otimes^2 A) \otimes \mathbb{Z}/2$
7	4	0	$\mathbb{Z}/2 * A$
8	4	$\mathbb{Z}^\Gamma \oplus \mathbb{Z}/3$	$\mathbb{Z}/3 \otimes A \oplus \Gamma(A)$
9	4	0	$\mathbb{Z}/3 * A \oplus R(A)$
9	5	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	$(\mathbb{Z}/2 \oplus \mathbb{Z}/3) \otimes A$
10	5	$\mathbb{Z}^A$	$(\mathbb{Z}/2 \oplus \mathbb{Z}/3) * A \oplus \Lambda^2(A)$

which is the union of two natural short exact sequences. By Remark 2.10(4) this shows that there are natural isomorphisms $A \otimes'' \mathbb{Z}^A \cong S^2(A) \cong A \otimes \mathbb{Z}^S$.

Remark 10.5. Decker obtained a formula for $H_m K(A, n)$, $m < 3n$, in terms of a list of homology operations α, see [14, Chapter III, (4.3)]. This list of homology operations (based on results of Cartan [11]) allows in principle the computation of $H_m K(A, n)$ as a functor and hence we can derive the quadratic $\mathbb{Z}$-module $H_m\{n\}$. The exact sequence of Theorem 10.3 still is helpful for understanding the somewhat intricate functors Ω_q and R_q which appear in Decker's formula. They generalize the functors Ω and R of Eilenberg–Mac Lane [16], that is $\Omega_0 = \Omega$, $R_0 = R$.

We now describe a connection between homotopy groups of Moore spaces and homology groups of Eilenberg–Mac Lane spaces. To this end recall that the Hurewicz homomorphism h is embedded in a long exact sequence [37]

$$\to H_{n+1}X \xrightarrow{b} \Gamma_n X \xrightarrow{i} \pi_n X \xrightarrow{h} H_n X \xrightarrow{b} \Gamma_{n-1} X \to$$

which is natural for simply connected spaces X. For an abelian group A we have the canonical map $(n \geq 2)$

$$k: M(A, n) \to K(A, n)$$

which induces the identity $H_n(k) = 1_A$ of A. This map induces the natural homomorphism

$$Q_1 = b^{-1}\Gamma_m(k)i^{-1}: \pi_m M(A, n) \to H_{m+1} K(A, n) \tag{10.4}$$

where we use i and b in the exact sequence above. Whitehead [37] showed that Q_1 is an isomorphism for $m = n + 1$. In the metastable range Q_1 is part of the following commutative diagram where we use $\Sigma M(A, n-1) = M(A, n)$, $m < 3n - 2$.

$$\begin{array}{ccccc}
\pi_m M(A,n) & \xrightarrow{H} & \pi_m M(A,n) \wedge M(A,n-1) & \xrightarrow{P} & \pi_m M(A,n) \\
\downarrow Q_1 & & \downarrow Q_2 & & \downarrow Q_1 \\
H_{m+1} K(A,n) & \xrightarrow{H} & H_{m+1} K(A,n) \wedge K(A,n) & \xrightarrow{P} & H_{m+1} K(A,n)
\end{array} \tag{10.5}$$

The maps H and P are defined as in (8.4) and (8.5) respectively. The map Q_2 is defined by $Q_2 = h\pi_{m+1}(k \wedge k)\Sigma$ where Σ is the suspension operator and where h is the Hurewicz map. Whence Q_2 is an isomorphism for $m = 2n - 1$. The commutativity of the diagram shows that $Q = (Q_1, Q_2)$ is a map between quadratic $\mathbb{Z}$-modules. We obtain the commutativity of (10.5) by the homotopy commutativity of

$$\begin{tikzcd} M(A,n) \arrow[r, "\mu'"] \arrow[d, "k"] & M(A,u) \vee M(A,n) \arrow[r, "\nabla"] \arrow[d, "k'"] & M(A,n) \arrow[d, "k"] \\ K(A,n) \arrow[r, "\Delta"] & K(A,n) \times K(A,n) \arrow[r, "\mu"] & K(A,n) \end{tikzcd} \tag{10.6}$$

Here μ' and μ are the comultiplication and multiplication respectively and k' is given by $k \vee k$ and the inclusion. By applying the functor Γ_m to (10.6) we essentially get (10.5).

For any $(n-1)$-connected space X with $H_n X \cong A$ we have maps

$$k: M(A,n) \xrightarrow{k'} X \xrightarrow{k''} K(A,n) \tag{10.7}$$

which induce isomorphisms in homology H_n. Here the homotopy class of k'' is unique, the homotopy class of k', however, is not unique. From (10.5) we derive for $m < 3n-2$ the commutative diagram

$$\begin{tikzcd} A \otimes \pi_m\{S^n\} \arrow[rr] \arrow[d, "\lambda"] & & A \otimes H_{m+1}\{n\} \arrow[d, "\lambda"] \\ \pi_m M(A,n) \arrow[r, "k'_* i^{-1}"] \arrow[rr, bend right, "Q_1"'] & \Gamma_m X \arrow[r, "b^{-1}k''_*"] & H_{m+1}K(A,n) \end{tikzcd} \tag{10.8}$$

which shows that $\Gamma_m X$ is non-trivial if Q_1 is non-trivial. The following lemma gives information on part of the kernel of Q_1.

Lemma 10.6. *Let $\alpha \in \pi_m(M(A,n))$ be a map which admits a factorization $\alpha: S^m \to Y \to M(A,n)$ where Y is n-connected and $\dim(Y) \leq m-1$. Then we have $Q_1(\alpha) = 0$. In particular, we have $Q_1([\xi, \eta]) = 0$ for all Whitehead products $[\xi, \eta]$ with $\xi \in \pi_t M(A,n)$, $t > n$.* □

Example 10.7. All arrows in (10.8) are isomorphisms for $n = 2$, $m = 3$. Moreover, the map

$$Q_1: \pi_4 M(A,2) \to H_5 K(A,2) \cong R(A)$$

is surjective and its kernel is the subgroup S in Theorem 9.3. Hence we have the natural isomorphisms

$$A *' \mathbb{Z}^{\Gamma} \cong {}_{\lambda}\pi_4 M(A, 2) \cong H_5 K(A, 2) \cong R(A),$$

compare Corollary 9.4.

11. Cohomology of Eilenberg–Mac Lane spaces

Here we obtain a six-term exact sequence for the cohomology groups of Eilenberg–Mac Lane spaces in the metastable range.

Let $\mathscr{R} \subset \mathscr{H}\mathscr{A}_n$ be a small ringoid, see Definition 8.1. A homotopy abelian H-space gives us the $\mathscr{R}^{\mathrm{op}}$-module $[\mathscr{R}, U]'$ as in (10.1). Now the quadratic $\mathscr{R}^{\mathrm{op}}$-modules $H^m\{\mathscr{R}, G\}$ and $\pi_K^m\{\mathscr{R}\}$ associated to the functors (8.3c) and (8.3b), respectively, yield the natural homomorphisms ($m < 3n$, resp. $\mathrm{hodim}(\Omega^m K) < 3n$)

$$\begin{aligned} &\lambda : H^m(U, G) \to \mathrm{Hom}_{\mathscr{R}^{\mathrm{op}}}([\mathscr{R}, U]', H^m\{\mathscr{R}, G\}), \\ &\lambda : \pi_K^m(U) \to \mathrm{Hom}_{\mathscr{R}^{\mathrm{op}}}([\mathscr{R}, U]', \pi_K^m\{\mathscr{R}\}). \end{aligned} \tag{11.1}$$

Compare (5.1). By Proposition 5.5 we have the following:

Proposition 11.1. *The homomorphisms λ in* (11.1) *are isomorphisms if $U = X_1 \times \cdots \times X_r$ is a finite product with $X_i \in \mathscr{R}$ for $i = 1, \ldots, r$ and if $\mathscr{R}$ is a full subringoid of $\mathscr{H}\mathscr{A}_n$.* □

Remark 11.2. Let $\mathscr{R}$ be the ringoid of elementary Eilenberg–Mac Lane spaces as in Remark 10.2. Then (11.1) yields the natural homomorphism

$$\lambda : \pi_K^m(K(A, n), G) \to \mathrm{Hom}_{\mathscr{R}^{\mathrm{op}}}([\mathscr{R}, K(A, n)], \pi_K^m\{\mathscr{R}\})$$

which is an isomorphism if A is finitely generated.

We now consider a special case of λ in (11.1). For this let $\mathscr{R} \cong \mathbb{Z}$ be the full subcategory consisting only of $K(\mathbb{Z}, n)$ and let $U = K(A, n)$. Then $H^m\{\mathscr{R}, G\}$ and $\pi_K^m\{\mathscr{R}\}$ are the quadratic $\mathbb{Z}$-modules

$$H_G^m\{n\} = \left(H^m(K(\mathbb{Z}, n), G) \xrightarrow{H} H^m(K(\mathbb{Z}, n) \wedge k(\mathbb{Z}, n), G) \right.$$
$$\left. \xrightarrow{P} H^m(K(\mathbb{Z}, n), G) \right),$$

$$\pi_K^m\{n\} = \left(\pi_K^m K(\mathbb{Z}, n) \xrightarrow{H} \pi_K^m K(\mathbb{Z}, n) \wedge K(\mathbb{Z}, n) \xrightarrow{P} \pi_K^m K(\mathbb{Z}, n) \right)$$

respectively defined as in (8.5). Now (11.1) yields the natural homomorphisms

$$\begin{aligned}&\lambda: H^m(K(A,n),G)\to \mathrm{Hom}_{\mathbb{Z}}(A, H_G^m\{n\}),\\ &\lambda: \pi_K^m(K(A,n))\to \mathrm{Hom}_{\mathbb{Z}}(A, \pi_K^m\{n\}),\end{aligned} \tag{11.2}$$

which are isomorphisms if A is a free abelian group (here A need not to be finitely generated). In the next result we use the derived functors in (7.4).

Theorem 11.3. *Let $m \le 3n-2$. Then there is a natural map $\kappa: \mathrm{Ext}(A, H_G^m\{n\}) \to H^m(K(A,n-1),G)$ such that ${}_\kappa H^m(K(A,n-1),G) = \mathrm{cok}(\kappa)$ is embedded in the natural exact sequence*

$$0\to \mathrm{Hom}'(A, H_G^m\{n\})\to {}_\kappa H^{m-1}(K(A,n-1),G)\to \mathrm{Hom}''(A, H_G^{m+1}\{n\})$$

$$\to \mathrm{Ext}(A, H_G^m\{n\}) \xrightarrow{\kappa} H^m(K(A,n-1),G) \xrightarrow{q} {}_\kappa H^m(K(A,n-1),G)\to 0,$$

where q is the quotient map. □

In the stable range $m < 2n-2$ this sequence is equivalent to the short exact sequence

$$0\to \mathrm{Ext}(A, H_G^m\{n\})\to H^m(K(A,n-1),G)\to \mathrm{Hom}(A, H_G^{m+1}\{n\})\to 0 \tag{11.3}$$

where $H_G^m\{n\} = H^m(K(\mathbb{Z},n),G)$ is an abelian group. Theorem 11.3 is a special case of the next result.

Theorem 11.4. *Let $\mathrm{hodim}(\Omega^m K) \le 3n-2$. Then there is a natural map $\kappa: \mathrm{Ext}(A, \pi_K^m\{n\}) \to \pi_K^m K(A,n-1)$ such that ${}_\kappa\pi_K^m K(A,n-1) = \mathrm{cok}(\kappa)$ is embedded in the natural exact sequence*

$$0\to \mathrm{Hom}'(A, \pi_K^m\{n\})\to {}_\kappa\pi_K^{m+1}K(A,n-1)\to \mathrm{Hom}''(A, \pi_K^{m-1}\{n\})$$

$$\to \mathrm{Ext}(A, \pi_K^m\{n\}) \xrightarrow{\kappa} \pi_K^m K(A,n-1) \xrightarrow{q} {}_\kappa\pi_K^m K(A,n-1)\to 0,$$

where q is the quotient map.

Again it is obvious how to describe the "cross effect sequence" of Theorem 11.4 by the formulas in Proposition 7.3.

Proof. The theorem is a special case of (3.7) in [8]. For this let g be given as in the proof of Theorem 10.3 with $P_g = K(A, n-1)$. Using the isomorphism λ in (11.2) (where we replace A by X_0 and X_1 respectively) we get the isomorphisms

$$H_0\{\pi_K^m\}(g^{\mathrm{op}}) = \mathrm{Ext}(A, \pi_K^m\{n\}),$$

$$H_1\{\pi_K^m\}(g^{\mathrm{op}}) = \mathrm{Hom}'(A, \pi_K^m\{n\}),$$

$$H_2\{\pi_K^m\}(g^{\mathrm{op}}) = \mathrm{Hom}''(A, \pi_K^m\{n\}).$$

Compare the definition in (7.4). Now i in [8, (3.7)] yields the homomorphism κ in Theorem 11.4. Therefore, Theorem 11.4, and also Theorem 11.3, is just a special case of [8, (3.7)]. □

Appendix A. Quadratic derived functors

In this appendix we associate with a quadratic $\mathscr{R}$-module M a chain functor and a cochain functor. If we apply these functors to a projective (resp. injective) resolution we get the quadratic derived functors which coincide with the classical derived functors in case $M_{ee} = 0$. We understand that Dold and Puppe [15] obtained derived functors of non-additive functors which as well generalized the classical derived functors of an additive functor; the construction of the quadratic derived functors below is different and relies on the structure of a quadratic module.

Let $\mathscr{R}$ be a ringoid with a zero object. An $\mathscr{R}$-module M yields the following *chain functors* which are as well denoted by M

$$M : \mathscr{R}_*/\simeq \;\to \mathscr{A}b_*/\simeq \quad \text{and} \quad M : \mathscr{R}^*/\simeq \;\to \mathscr{A}b^*/\simeq \;; \tag{A.1}$$

compare the notation in (6.1). For a chain complex X_* in $\mathscr{R}_*$ we define $M(X_*)$ simply by setting $M(X_*)_n = M(X_n)$. The differential d_* in $M(X_*)$ is induced by the differential d in X_*, $d_* = M(d)$. Similarly we get induced chain maps $M(F)$ with $M(F)_n = M(F_n)$ and induced chain homotopies $M(\alpha)$ with $M(\alpha)_n = M(\alpha_n)$. Since M is an additive functor one readily observes that this chain functor is well defined. In the same way one gets the cochain functor M which carries $X^* \in \mathscr{R}^*$ to the cochain complex $M(X^*)$.

Now let M be a quadratic $\mathscr{R}$-module. We associate with M the *quadratic chain functors* M as in (A.1) which again are simply denoted by M, see Definitions A.1 and A.2. In fact, if $M_{ee} = 0$ these chain functors coincide with the additive functors above.

Definition A.1. For X_* in $\mathscr{R}_*$ the chain complex $C_* = M(X_*)$ is given by the abelian groups $(n \geq 2)$

$$(1) \qquad \begin{aligned} C_0 &= M(X_0), \\ C_1 &= \operatorname{cok}\{(P, -(1,d)_*): M(X_1, X_1) \to M(X_1) \oplus M(X_1, X_0)\}, \\ C_n &= \operatorname{cok}\{P \oplus (1,d)_*: M(X_n, X_n) \oplus M(X_n, X_1) \to M(X_n) \oplus M(X_n, X_0)\}. \end{aligned}$$

The differential $d = d_n: C_n \to C_{n-1}$ is induced by the maps

$$(2) \qquad d_1 = (d_*, P(d,1)_*), \qquad d_n = d_* \oplus (d,1)_*, \quad n \geq 2.$$

For a chain map $F: X_* \to Y_*$ we get the induced chain map $M(F): MX_* \to MY_*$ by

$$(3) \qquad (MF)_0 = (F_0)_*, \qquad (MF)_n = (F_n)_* \oplus (F_n, F_0)_*, \quad n \geq 1.$$

Finally a chain homotopy $\alpha: F \simeq G$, $\alpha_n: X_{n-1} \to Y_n$ in $\mathscr{R}_*$ yields a chain homotopy $M\alpha: MF \simeq MG$ by

$$(4) \qquad (M\alpha)_1 = ((\alpha_1)_*, (\alpha_1, F_0)_* H), \qquad (M\alpha)_n = (\alpha_n)_* \oplus (\alpha_n, F_0)_*, \quad n \geq 2.$$

The next definition is dual to Definition A.1.

Definition A.2. For X^* in $\mathscr{R}^*$ the cochain complex $C^* = MX^*$ is given by the abelian groups $(n \geq 2)$

$$(1) \qquad \begin{aligned} C^0 &= M(X^0) \\ C^1 &= \ker\{(H, -(1,d)_*): M(X^1) \oplus M(X^1, X^0) \to M(X^1, X^1)\} \\ C^n &= \ker\{H \oplus (1,d)_*: M(X^n) \oplus M(X^n, X^0) \to M(X^n, X^n) \oplus M(X^n, X^0)\} \end{aligned}$$

The differential $d = d^n: C^n \to C^{n+1}$ is induced by the maps

$$(2) \qquad d^1 = (d_*, (d,1)_* H), \qquad d^n = d_* \oplus (d,1)_*, \quad n \geq 2.$$

For a chain map $F: X^* \to Y^*$ we get the induced chain map $M(F): MX^* \to MY^*$ by

$$(3) \qquad (MF)^0 = (F^0)_*, \qquad (MF)^n = (F^n)_* \oplus (F^n, F^0)_*, \quad n \geq 1.$$

Finally a chain homotopy $\alpha: F \simeq G$ $(\alpha^n: X^{n+1} \to Y^n)$ in $\mathscr{R}^*$ yields a chain homotopy $M\alpha: MF \simeq MG$ by

$$(4) \qquad (M\alpha)^0 = ((\alpha^0)_*, P(\alpha^0, F^0)_*), \qquad (M\alpha)^n = (\alpha^n)_* \oplus (\alpha^n, F^0)_*, \quad n \geq 1.$$

Proposition A.3. *Definitions* A.1 *and* A.2 *yield well-defined functors* $M:\mathscr{R}_*/\simeq\ \to \mathscr{Ab}_*/\simeq$ *and* $M:\mathscr{R}^*/\simeq\ \to \mathscr{Ab}^*/\simeq$, *respectively.* $\square$

The functors M in Proposition A.3 are quadratic, the cross effect of these functors is described below. The proof of Proposition A.3 is similar to the proof of Theorem 6.2, in fact, Theorem 6.2 can be used for the 1-dimensional part of the proposition, compare Remark A.4.

We point out that the definition of the quadratic chain functors relies on the structure maps H and P of the quadratic $\mathscr{R}$-module M; a functor $\mathscr{R}\to\mathscr{Ab}$ which is merely quadratic is not appropriate for the definition of the functors in Proposition A.3.

Remark A.4. The quadratic chain functors M_* and M^* in Definition 6.1 are related to the quadratic chain functors M in Proposition A as follows. Let $d_1:X_1\to X_0$ and $d^0:X^0\to X^1$ be given by X_* and X^* respectively. Then the 1-dimensional part of MX_*, resp. of MX^*, coincides with the map

$$M_1(d_1)/\text{boundaries}\to M_0(d_1),\quad \text{resp. } M^0(d^0)\to \text{cycles}\subset M^1(d^0),$$

compare Definitions 6.1, A.1 and A.2. This shows that (with $X_i=0, X^i=0, i\geq 2$) one has isomorphic homology groups $H_iMX_*=H_iM_*(d_1)$, $H^iMX^*=H^iM^*(d^0)$ for $i=0,1$. The homology $H_2M_*(d_1)$ and $H^2M^*(d^0)$, however, cannot be obtained by MX_* and MX^* respectively.

We now assume that the additive category $\mathscr{A}$ is an abelian category with enough projectives and injectives respectively, for example $\mathscr{A}=\mathscr{M}(\mathscr{R})$. The homology of chain complexes in $\mathscr{A}$ is defined. We say that X_* is a *projective resolution* of $X\in\mathrm{Ob}(\mathscr{A})$ if a chain map $\varepsilon:X_*\to X$ in $\mathscr{A}_*$ is given (which induces an isomorphism in homology) where all X_i of X_* are projective in $\mathscr{A}$ and where X is the chain complex concentrated in degree 0. On the other hand X^* is an *injective resolution* of X if a chain map $\varepsilon:X\to X^*$ in $\mathscr{A}^*$ is given (which induces an isomorphism in cohomology) where all X^i of X^* are injective in $\mathscr{A}$. It is well known that the choice of resolutions X_*, X^* yields functors $i:\mathscr{A}\to\mathscr{A}^*/\simeq$ and $j:A\to A_*/\simeq$ which are well defined up to canonical isomorphisms.

Definition A.5. Let $\mathscr{A}$ be an abelian category as above and let $M:\mathscr{A}\to\mathscr{Ab}$ be a quadratic functor. Then Example 3.4 shows that M yields a quadratic $\mathscr{A}$-module $M=M\{A\}$ as well denoted by M. Using the resolution functors i,j above and using Proposition A.3 one gets functors

(1) $\quad Mi:\mathscr{A}\to\mathscr{Ab}_*/\simeq\quad$ and $\quad Mj:\mathscr{A}\to\mathscr{Ab}^*/\simeq$.

The nth (co)homology of these functors yields the *quadratic derived functors* $L_nM : \mathscr{A} \to \mathscr{Ab}$, $R^nM : \mathscr{A} \to \mathscr{Ab}$ respectively, $n \geq 0$. For $X \in \mathrm{Ob}(\mathscr{A})$ one has

$$(2) \qquad (L_nM)X = H_nMX_* \quad \text{and} \quad (R^nM)X = H^nMX^*$$

where X_*, X^* are resolution as above. The chain complexes MX_*, MX^* are defined as in Definitions A.1 and A.2.

Remark A.6. In case M in Definition A.5 is an additive functor, that is $M_{ee} = 0$, the derived functors coincide with the classical derived functors of M, see for example [12, 18]. For a quadratic functor M Dold and Puppe [15] as well defined derived functors; their construction, however, is different to the one in Defintion A.5 and is available for any non-additive functor $\mathscr{A} \to \mathscr{Ab}$, Remarks 6.3 and 7.1. Our definition in Definition A.5 is adapted especially to quadratic functors. In degree $n = 0,1$ the derived functors above coincide with the derived functors of Dold-Puppe.

Definition A.7. Let $\mathscr{A}$ be an abelian category and let $M : \mathscr{A} \to \mathscr{Ab}$ be a quadratic functor. We say that M is *quadratic right exact* if each exact sequence $X_1 \xrightarrow{d} X_0 \xrightarrow{q} X \to 0$ in $\mathscr{A}$ induces an exact sequence

$$M(X_1) \oplus M(X_1 \mid X_0) \xrightarrow{(d_*,\, P(d,\,1)_*)} M(X_0) \xrightarrow{q_*} M(X) \to 0.$$

We say that M is *quadratic left exact* if each exact sequence $0 \to X \xrightarrow{i} X^0 \xrightarrow{d} X^1$ in $\mathscr{A}$ induces an exact sequence

$$0 \to M(X) \xrightarrow{i_*} M(X^0) \xrightarrow{(d_*,\, (d,\,1)_*H)} M(X^1) \oplus M(X^1 \mid X^0).$$

The definitions immediately imply as in the classical case:

Lemma A.8. *Let $M : \mathscr{A} \to \mathscr{Ab}$ be quadratic right exact. Then one has the natural isomorphism $M \cong L_0M$. Dually, if M is quadratic left exact one has the natural isomorphism $M \cong R^0M$.* $\square$

As examples of quadratic derived functors we obtain the following *quadratic* Tor *and* Ext *functors* for a small ringoid $\mathscr{R}$, $n \geq 0$:

$$\begin{aligned} &\mathrm{Tor}_n^{\mathscr{R}} : \mathscr{M}(\mathscr{R})^{\mathrm{op}} \times \mathscr{Q}\mathscr{M}(\mathscr{R}) \to \mathscr{Ab}, \\ &\mathrm{Ext}_{\mathscr{R}}^n : \mathscr{M}(\mathscr{R})^{\mathrm{op}} \times \mathscr{Q}\mathscr{M}(\mathscr{R}) \to \mathscr{Ab}. \end{aligned} \qquad \text{(A.2)}$$

For M in $\mathscr{QM}(\mathscr{R})$ these functors are derived from the quadratic functors

(1) $$-\otimes_{\mathscr{R}} M : \mathscr{M}(\mathscr{R}^{\mathrm{op}}) \to \mathscr{Ab},$$

(2) $$\mathrm{Hom}_{\mathscr{R}}(-, M) : \mathscr{M}(\mathscr{R})^{\mathrm{op}} \to \mathscr{Ab},$$

that is, for a projective resolution X_* of X in $\mathscr{M}(\mathscr{R}^{\mathrm{op}})$ and for a projective resolution Y_* of Y in $\mathscr{M}(\mathscr{R})$ we set

(3) $$\mathrm{Tor}_n^{\mathscr{R}}(X, M) = L_n(-\otimes_{\mathscr{R}} M)(X),$$

(4) $$\mathrm{Ext}_{\mathscr{R}}^n(Y, M) = R^n\mathrm{Hom}_{\mathscr{R}}(-, M)(Y).$$

In (4) we consider Y_* as an injective resolution in $\mathscr{M}(\mathscr{R})^{\mathrm{op}}$ and we use (8.2). Clearly the groups (3), (4) are trivial for $n \geq 1$ in case X and Y are projective objects in $\mathscr{A}$. The functors (A.2) are quadratic in the first variable and additive in the second variable.

Proposition A.9. *The functor* $-\otimes_{\mathscr{R}} M$ *is quadratic right exact and the functor* $\mathrm{Hom}_{\mathscr{R}}(-, M)$ *is quadratic left exact so that we have natural isomorphisms (see Lemma* A.8).

$$\mathrm{Tor}_0^{\mathscr{R}}(X, M) = X \otimes_{\mathscr{R}} M \quad \text{and} \quad \mathrm{Ext}_{\mathscr{R}}^0(Y, M) = \mathrm{Hom}_{\mathscr{R}}(Y, M).$$

In case M is an $\mathscr{R}$-module, that is $M_{ee} = 0$, the Tor and Ext groups above coincide with the classical groups, see [18].

Example A.10. Let $\mathscr{R} = \mathbb{Z}$ be the ring of integers and let M be a quadratic $\mathbb{Z}$-module. For an abelian group A one gets (see (7.4)) $\mathrm{Tor}_1^{\mathbb{Z}}(A, M) = A *' M$ and $\mathrm{Ext}_{\mathbb{Z}}^1(A, M) = \mathrm{Ext}'(A, M)$. This follows since d_A in (7.4) is a projective resolution of A, see Remark A.4. Clearly $\mathrm{Tor}_n^{\mathbb{Z}} = 0 = \mathrm{Ext}_{\mathbb{Z}}^n$ for $n \geq 2$ since the chain complex d_A is 1-dimensional.

Appendix B. The cross effect of quadratic derived functors

We introduce biderived functors which describe the cross effects of the quadratic derived functors above. Moreover, we discuss various exact sequences for these functors. We assume that $\mathscr{R}$ is a ringoid with a zero object.

Definition B.1. Let M be an $\mathscr{R} \times \mathscr{R}$-module, see Definition 1.2. Then we define the additive functor

$$M : \mathscr{R}_*/\simeq \otimes \mathscr{R}_*/\simeq \to \mathscr{Ab}_*/\simeq$$

(as well denoted by M) as follows. For chain complexes X_*, Y_* in $\mathscr{R}_*$ we get $C_* = M(X_*, Y_*)$ by $(n \geq 2)$

(1) $$C_0 = M(X_0, Y_0),$$
$$C_1 = \operatorname{cok}\{((1, d)_*;\ -(d, 1)_*): M(X_1, Y_1) \to M(X_1, Y_0) \oplus M(X_0, Y_1)\},$$
$$C_n = \operatorname{cok}\{(1, d)_* \oplus (d, 1)_*: M(X_n, Y_1) \oplus M(X_1, Y_n) \to M(X_n, Y_0) \oplus M(X_0, Y_n)\}.$$

The differential $d = d_n: C_n \to C_{n-1}$ is induced by the maps

(2) $$d_1 = ((d, 1)_*, (1, d)_*), \qquad d_n = (d, 1)_* \oplus (1, d)_*, \quad n \geq 2.$$

For chain maps $F: X_* \to X'_*$, $G: Y_* \to Y'_*$ we get the induced chain map $M(F \otimes G): M(X_*, Y_*) \to M(X'_*, Y'_*)$ by

(3) $$M(F \otimes G)_0 = (F_0, G_0)_*$$
$$M(F, G)_n = (F_n, G_0)_* \oplus (F_0, G_n)_*, \quad n \geq 1.$$

Finally, chain homotopies $\alpha: F \simeq F'$, $\beta: G \simeq G'$ yield a chain homotopy $M(\alpha, \beta): M(F \otimes G) \simeq M(F' \otimes G')$ by

(4) $$M(\alpha, \beta)_1 = ((\alpha_1, G_0)_*, (F_0, \beta_1)_*),$$
$$M(\alpha, \beta)_n = (\alpha_n, G_0)_* \oplus (F_0, \beta_n)_*, \quad n \geq 2.$$

The next definition is dual to Definition B.1.

Definition B.2. We associate with an $\mathscr{R} \otimes \mathscr{R}$-module M the additive functor

$$M: \mathscr{R}^*/\simeq\ \otimes \mathscr{R}^*/\simeq\ \to \mathscr{Ab}^*/\simeq$$

(as well denoted by M) as follows. For cochain complexes X^*, Y^* in $\mathscr{R}^*$ we get $C^* = M(X^*, Y^*)$ by $(n \geq 2)$

(1) $$C^0 = M(X^0, Y^0),$$
$$C^1 = \ker\{((1, d)_*, -(d, 1)_*): M(X^1, Y^0) \oplus M(X^0, Y^1) \to M(X^1, Y^1)\},$$
$$C^n = \ker\{(1, d)_* \oplus (d, 1)_*: M(X^n, Y^0) \oplus M(X^0, Y^n) \to M(X^n, Y^1) \oplus M(X^1, Y^n)\}$$

The differential $d = d_n : C^n \to C^{n+1}$ is induced by the maps

(2) $$d^1 = ((d,1)_*, (1,d)_*), \qquad d^n = (d,1)_* \oplus (1,d)_*, \quad n \geq 2.$$

For chain maps $F : X^* \to X'^*$, $G : Y^* \to Y'^*$ we get the induced chain map $M(F \otimes G) : M(X^*, Y^*) \to M(X'^*, Y'^*)$ by

(3) $$M(F \otimes G)^0 = (F^0, G^0)_*,$$
$$M(F, G)^n = (F^n, G^0)_* \oplus (F^0, G^n)_*, \quad n \geq 1.$$

Finally, chain homotopies $\alpha : F \simeq F'$, $\beta : G \simeq G'$ yield a chain homotopy $M(\alpha, \beta) : M(F \otimes G) \simeq M(F' \otimes G')$ by

(4) $$M(\alpha, \beta)^0 = ((\alpha^0, G^0)_*, (F^0, \beta^0)_*),$$
$$M(\alpha, \beta)^n = (\alpha^n, G^0)_* \oplus (F^0, \beta)_*, \quad n \geq 1.$$

As in (8.3) one can readily check the following:

Proposition B.3. *The functors in Definitions* B.1 *and* B.2 *are well defined and additive.* □

The crucial property of the functors in Definitions B.1 and B.2 is described by the next result.

Theorem B.4. *Let M be a quadratic $\mathscr{R}$-module and let $M(X_* \mid Y_*)$ and $M(X^* \mid Y^*)$ be cross effects of the quadratic functors M in Definitions* A.1 *and* A.2 *respectively. Then there are natural isomorphisms*

$$\Psi : M(X_* \mid Y_*) \cong M_{ee}(X_*, Y_*) \quad \text{and} \quad \chi : M_{ee}(X^*, Y^*) \cong M(X^* \mid Y^*)$$

of chain complexes. Here M_{ee} is the $\mathscr{R} \otimes \mathscr{R}$-module given by M, see Definitions 3.1 *and* 1.2, *and $M_{ee}(X_*, Y_*)$ and $M_{ee}(X^*, Y^*)$ are defined by Definitions* B.1 *and* B.2 *respectively.* □

Similarly as in Definition A.5 we can use the functors in Definitions B.1 and B.2 for the definitions of derived functors. Let $\mathscr{A}$ be an abelian category with enough projective and injectives.

Definition B.5. Let M be an $\mathscr{A} \otimes \mathscr{A}$-module. Using the resolution functors $i : \mathscr{A} \to \mathscr{A}_* / \simeq$ and $j : \mathscr{A} \to \mathscr{A}^* / \simeq$ one gets the additive functors

(1) $$M(i \otimes i) : \mathscr{A} \otimes \mathscr{A} \to \mathscr{Ab}_* / \simeq \quad \text{and} \quad M(j \otimes j) : \mathscr{A} \otimes \mathscr{A} \to \mathscr{Ab}^* / \simeq .$$

The nth (co)homology of these functors yields the *biderived functors*

$$L_n M : \mathscr{A} \otimes \mathscr{A} \to \mathscr{Ab} \quad \text{and} \quad R^n M : \mathscr{A} \otimes \mathscr{A} \to \mathscr{Ab},$$

respectively, $n \geq 0$. For $X, Y \in \mathrm{Ob}(\mathscr{A})$ one has

$$(2) \qquad (L_n M)(X, Y) = H_n M(X_*, Y_*), \qquad (R^n M)(X, Y) = H^n M(X^*, Y^*),$$

where X_*, Y_* (resp. X^*, Y^*) are projective (resp. injective) resolutions of X, Y. The chain complexes $M(X_*, Y_*)$, $M(X^*, Y^*)$ are defined in Definitions B.1 and B.2.

As a corollary of Theorem B.4 one gets immediately the following:

Corollary B.6. *Let M be a quadratic $\mathscr{A}$-module. Then the quadratic derived functors in Definition* A.5 *have the cross effects*

$$(L_n M)(X \mid Y) = (L_n M_{ee})(X, Y), \qquad (R^n M)(X \mid Y) = (R^n M_{ee})(X, Y),$$

where M_{ee} is the $\mathscr{A} \otimes \mathscr{A}$-module given by M. $\square$

In addition to Corollary B.6 one gets the following natural exact sequences for quadratic derived functors, they correspond to the classical exact sequences for derived functors in case $M_{ee} = 0$. To this end we consider a short exact sequence

$$S = \left(0 \to X \xrightarrow{i^s} Y \xrightarrow{q^s} Z \to 0 \right) \tag{B.1}$$

in $\mathscr{A}$ and maps $S \to S'$ between such sequences.

Theorem B.7. *Let M be a quadratic $\mathscr{A}$-module. Then S in* (B.1) *yields the following natural commutative diagram in which the rows and columns are long exact sequences* ($n \in \mathbb{Z}$):

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & & \\
 & L_{n+1} M_{ee}(X, Y) \arrow[d, "\partial"] & & L_n M_{ee}(X, Z) \arrow[d] & & \\
\to L_{n+1} M q^s \arrow[r, "\partial"] \arrow[d] & L_n M X \arrow[r, "i^s_*"] \arrow[d] & L_n M Y \arrow[r] \arrow[d, equal] & L_n M q^s \arrow[r, "\partial"] \arrow[d] & L_{n-1} M X \to \arrow[d] \\
\to L_{n+1} M Z \arrow[r, "\partial"] & L_n M i^s \arrow[r] \arrow[d] & L_n M Y \arrow[r, "q^s_*"] & L_n M Z \arrow[r, "\partial"] \arrow[d, "\partial"] & L_{n-1} M i^s \to \\
 & L_n M_{ee}(X, Z) \arrow[d] & & L_{n-1} M_{ee}(X, Z) \arrow[d] & \\
 & {} & & {} &
\end{tikzcd}$$

We leave it to the reader to write down the dual diagram for right derived functors R^n; for this we simply replace L_* by R^* in such a way that ∂ raises the degree by 1. If $M_{ee} = 0$ we see that the rows of the diagram are isomorphic, in this case the row coincide with the classical exact sequence for left derived functors, see [18, Chapter IV, Section 6]. In case the sequence S is split all boundaries ∂ are trivial and the remaining short exact sequences are split, this yields Corollary B.6.

Proof of Theorem B.7. We can choose a short exact sequence of projective resolutions

$$(1) \qquad 0 \to X_* \xrightarrow{i} Y_* \xrightarrow{r} Z_* \to 0$$

of S, compare the proof of [18, Chapter IV, 6.1]. As a module we have $Y_n = X_n \oplus Z_n$. The differential of Y_* is given by

$$(2) \qquad (d \oplus d) + i_1 \xi r_2 : X_n \oplus Z_n = Y_n \to X_{n-1} \oplus Z_{n-1} = Y_{n-1}.$$

Here d denotes the differential of X_* and Y_* respectively. We now derive from (1) the following commutative diagram in which rows and columns are short exact sequences of chain complexes:

$$(3) \qquad \begin{tikzcd} & & M(X_*, Z_*) \arrow[d, tail, "j"] \\ MX_* \arrow[r, tail, "i_*"] \arrow[d, tail] & MY_* \arrow[r, two heads] \arrow[d, equal] & \mathrm{cok}(i_*) \arrow[d, two heads] \\ \ker(q_*) \arrow[r, tail] \arrow[d, two heads] & MY_* \arrow[r, two heads, "q_*"'] & MZ_* \\ M(X_*, Z_*) & & \end{tikzcd}$$

The maps j are well-defined chain maps since we have (2) for the boundary in Y_*. We now set

$$(4) \qquad L_n M i^s = H_n \ker(q_*), \qquad L_n M q^s = H_n \mathrm{cok}(i_*).$$

Now Theorem B.7 is obtained by the long exact sequences associated to short exact sequences of chain complexes. $\square$

There are the following examples of biderived functors. We associate with M in $\mathscr{M}(\mathscr{R} \otimes \mathscr{R})$ the additive functors

$$\begin{aligned} &- \otimes_{\mathscr{R} \otimes \mathscr{R}} M : \mathscr{M}(\mathscr{R}^{\mathrm{op}}) \otimes \mathscr{M}(\mathscr{R}^{\mathrm{op}}) \to \mathscr{Ab}, \\ &\mathrm{Hom}_{\mathscr{R} \otimes \mathscr{R}}(-, M) : \mathscr{M}(\mathscr{R})^{\mathrm{op}} \otimes \mathscr{M}(\mathscr{R})^{\mathrm{op}} \to \mathscr{Ab}, \end{aligned} \tag{B.2}$$

which carry the object (X, Y) to $(X \otimes Y) \otimes_{\mathscr{R} \otimes \mathscr{R}} M$ and $\mathrm{Hom}_{\mathscr{R} \otimes \mathscr{R}}(X \otimes Y, M)$ respectively, compare Propositions 4.2(3) and 5.2(3). The biderived functors of (B.2) are denoted by

$$\text{(1)} \qquad \mathrm{Tor}_n^{\mathscr{R} \otimes \mathscr{R}}(X, Y, M) = L_n(- \otimes_{\mathscr{R} \otimes \mathscr{R}} M)(X, Y),$$

$$\text{(2)} \qquad \mathrm{Ext}^n_{\mathscr{R} \otimes \mathscr{R}}(X, Y, M) = R^n(\mathrm{Hom}_{\mathscr{R} \otimes \mathscr{R}}(-, M)(X, Y)).$$

Using Corollary 9.4 one obtains for a quadratic $\mathscr{R}$-module M the cross effects $(n \geq 0)$

$$\text{(3)} \qquad \mathrm{Tor}_n^{\mathscr{R}}(X \mid Y, M) = \mathrm{Tor}_n^{\mathscr{R} \otimes \mathscr{R}}(X, Y, M_{ee}),$$

$$\text{(4)} \qquad \mathrm{Ext}^n_{\mathscr{R}}(X \mid Y, M) = \mathrm{Ext}^n_{\mathscr{R} \otimes \mathscr{R}}(X, Y, M_{ee}).$$

As an example of (1) we get for $\mathscr{R} = \mathbb{Z}$ the triple torsion product of Mac Lane [21],

$$\text{(5)} \qquad \mathrm{Tor}_1^{\mathbb{Z}}(X, Y, M) = \mathrm{Trp}(X, Y, M) = H_1(d_X \otimes d_Y, M),$$

compare Proposition 7.3(3). We can also apply Theorem B.7 for the functors in (3) and (4); this leads for $\mathscr{R} = \mathbb{Z}$ to the following results on the functors in (7.4), see Example A.10.

Theorem B.9. *Let M be a quadratic $\mathbb{Z}$-module and let $S: 0 \to X \xrightarrow{i} Y \xrightarrow{q} Z \to 0$ be an exact sequence of abelian groups. Then one has the following commutative diagrams in which the rows and the rectangle sequences of broken arrows are exact sequences of abelian groups. Moreover, these diagrams are natural in S.*

$$\begin{tikzcd}
 & H_1(d_X \otimes d_Z, M_{ee}) \arrow[rrr, dashed, "\partial"] & & & X \otimes M \arrow[d] & & & \\
0 \arrow[r] & i *' M \arrow[r] \arrow[u] \arrow[d] & Y *' M \arrow[r, "q_*"] & Z *' M \arrow[r, "\partial"] & i \otimes M \arrow[r] \arrow[d] & Y \otimes M \arrow[r, "q_*"] & Z \otimes M \arrow[r] & 0 \\
 & X *' M & & 0 \arrow[ll, dashed] & X \otimes Z \otimes M_{ee} \arrow[l, dashed] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & H^1(d_X \otimes d_Z, M_{ee}) \arrow[d] & & & \mathrm{Hom}(X, M) \arrow[lll, dashed, "\partial"'] & & & \\
0 & \mathrm{Ext}(i, M) \arrow[l] \arrow[d] & \mathrm{Ext}(Y, M) \arrow[l] & \mathrm{Ext}(Z, M) \arrow[l, "q^*"'] & \mathrm{Hom}(i, M) \arrow[l, "\partial"'] \arrow[u] & \mathrm{Hom}(Y, M) \arrow[l] & \mathrm{Hom}(Z, M) \arrow[l, "q_*"'] & 0 \arrow[l] \\
 & \mathrm{Ext}(X, M) \arrow[rr, dashed] & & 0 \arrow[r, dashed] & \mathrm{Hom}(X \otimes Z, M_{ee}) \arrow[u] & & &
\end{tikzcd}$$

In case $M_{ee} = 0$ the diagram above corresponds exactly to the classical six-term exact sequences. We can apply these exact sequences for example if M is the quadratic $\mathbb{Z}$-module $M = \mathbb{Z}^r$. In this case the torsion product $Y *' \mathbb{Z}^r = R(Y)$ corresponds to the functor R of Eilenberg and Mac Lane, see Remark 10.7.

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